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Exercises:

1. Solve Exercise 23.D.6 in MWG.

2. (Variant of Exercise 23.D.5 in MWG)
   Consider a *sealed-bid all-pay auction* in which every buyer submits a bid, the highest bidder receives the good (with symmetric tie-breaking), and *every* buyer pays the seller the amount of his bid regardless of whether he wins.

   Suppose there are I symmetric buyers and each buyer's valuation is independently drawn from the interval $[\underline{\theta}, \overline{\theta}] \subset \mathbb{R}_+$ according to a stricly positive density.

   (a) Argue that any symmetric pure strategy equilibrium of this auction yields the seller the same expected revenue as the sealed-bid second-price auction.

   (b) Show that this auction indeed has a symmetric equilibrium in pure strategies.
   [*Hint:* Try to construct an explicit equilibrium using the payoff equivalence result.]

You can use the following envelope theorem in the next exercise. We will discuss the idea of the theorem and its proof in the tutorial.

Theorem (Milgrom and Segal, 2002)**.**
Let X be an arbitrary set, $T = [\underline{t}, \overline{t}]$,[1] and $f : X \times T \to \mathbb{R}$. Denote

$$V(t) = \sup_{x \in X} f(x,t) \tag{1}$$

$$X^*(t) = \{x \in X | f(x,t) = V(t)\}. \tag{2}$$

Suppose that $f(x, \cdot)$ is differentiable for all $x \in X$, $f_t(x, \cdot)$ is uniformly bounded and that $X^*(t) \neq \emptyset$ for almost all t. Then for any selection $x^*(t) \in X^*(t)$,

$$V(t) = V(\underline{t}) + \int_{\underline{t}}^{t} f_t(x^*(s), s) ds. \tag{3}$$

3. *(advanced)* Consider the general mechanism design setting from the lecture, where $v_i(k, \theta_i)$ denotes the value of allocation k to agent i with type θ_i. Suppose that $\Theta_i = [\underline{\theta}_i, \overline{\theta}_i] \subset \mathbb{R}$ and that v_i is differentiable in θ_i for all k and the derivative is uniformly bounded. Given a direct revelation mechanism (k, t), let $U_i(\theta) = v_i(k(\theta), \theta_i) + t_i(\theta)$ be the utility of agent i if θ is the profile of types and all agents report truthfully.

   (a) Show that if the direct revelation mechanism (k, t) is implementable in dominant strategies, then

   $$U_i(\theta) = U_i(\underline{\theta}_i, \theta_{-i}) + \int_{\underline{\theta}_i}^{\theta_i} \frac{\partial v_i(k(s, \theta_{-i}), s)}{\partial \theta_i} ds. \tag{ICFOC}$$

   Suppose that $v_i(k, \theta_i)$ has the single-crossing property: $\frac{\partial^2 v_i(k, \theta_i)}{\partial k \, \partial \theta_i}$ exists and is stricly positive for all $k \in K$ and $\theta_i \in \Theta_i$.

   (b) Show that if the direct revelation mechanism (k, t) is implementable in dominant strategies, then $k(\theta_i, \theta_{-i})$ is weakly increasing in θ_i for all θ_{-i}.

   (c) Show that any monotone mechanism that satisfies (ICFOC) is implementable in dominant strategies.

   (d) Discuss the relation of these results to the result that you saw in the lecture.

   (e) Show: If a direct revelation mechanism implements the value-maximizing allocation rule in dominant strategies, then it is a VCG mechanism.

[1] This result holds more generally, for example if $T \subset \mathbb{R}^n$ is convex.