Exercises:

1. Solve Exercise 21.D.5 in MWG.

2. Solve Exercise 21.D.10 in MWG.

3. Read Section 8.4 in Börgers (2014) and solve Exercise 8.6.e.
   Let me know if you do not have a copy of the book.

4. Gibbard-Satterthwaite (harder)
   Consider the Gibbard-Satterthwaite setting from the lecture: \( X \) denotes the set of alternatives, \( |X| \geq 3 \), and \( R_i = \mathcal{P} \) for all agents \( i \), where \( \mathcal{P} \) denotes the set of all rational preference profiles without indifferences.

   Given a preference \( P_i \) of agent \( i \), denote by \( r_1(P_i) \) the alternative that is ranked first under preference \( P_i \) (and analogously for \( r_k(P_i) \)).

   (a) Show: Suppose the social choice function \( f \) is dominant-strategy incentive compatible. Then \( f \) is monotonic.

   (b) Show: Suppose the social choice function \( f \) is dominant-strategy incentive compatible and onto.
      Then \( f \) is Pareto efficient.

Suppose from now on that there are only two agents, 1 and 2, and consider a social choice function \( f \) that is dominant-strategy incentive compatible and onto.

(c) Let \( x, y \in X \), \( x \neq y \) and consider a preference profile \( (P_1, P_2) \) such that \( r_1(P_1) = r_2(P_2) = x \), \( r_2(P_1) = r_1(P_2) = y \), and suppose that \( f(P_1, P_2, \cdot) = x \). Show that \( f(P_1, P_2) = x \) for all \( (P_1, P_2) \) such that \( r_1(P_1) = x \).

(d) Repeating the previous argument for all pairs of alternatives yields two sets \( X_1 \subseteq X \) and \( X_2 \subseteq X \) such that \( z \in X_1 \) if and only if \( r_1(P_1) = z \) implies \( f(P_1, P_2) = z \) (and \( z \in X_2 \) if and only if \( r_1(P_2) = z \) implies \( f(P_1, P_2) = z \)).
   Show that \( X_1 = X \). What do you conclude?
   (Hint: Use the previous part! Argue that \( X_0 = X \setminus \{X_1 \cup X_2\} \) contains at most one alternative and hence \( X_1 \cup X_2 \) contains at least two distinct alternatives. Argue then that either \( X_1 \) or \( X_2 \) must be empty. Using the previous part you can show now that \( X_0 \) must also be empty.)