Exercises:

1. **Interdependent value auction**

   Suppose there is one object for sale and \( N \) potential buyers. Each agent privately observes a signal \( X_i \), which is independently and identically distributed on \([0, X]\) with cdf \( F \) and density \( f \). Denote by \( G = F_{(1:N-1)} \) the cdf of the first-order statistic of \( N - 1 \) of these random variables.

   Buyers have quasi-linear utilities: in case of winning the object, buyer \( i \) gets utility \( v(x_i, x_{-i}) - p \), where \( p \) denotes the payment made, and he gets utility of 0 in case of not winning. Suppose that \( v \) is positive, strictly increasing in all signals, symmetric in the last \( N - 1 \) signals, and denote by \( v(x_i, y) \) the expected valuation of agent \( i \) given he received signal \( x_i \) and the highest signal among all other signals has value \( y \).

   (a) Show: In a second price auction, each agent bidding according to the bid function \( \beta(x_i) = v(x_i, x_i) \) is a Bayes-Nash equilibrium.

   Is it a dominant strategy to follow this bid function? Is it an ex-post equilibrium?

   (b) Consider an open English auction. A symmetric strategy in an English auction is a collection \( \beta = (\beta^N, \beta^{N-1}, \ldots, \beta^2) \) of \( N - 1 \) functions \( \beta^k : [0, X] \times \mathbb{R}^{N-k} \rightarrow \mathbb{R}_+ \). The interpretation is that \( \beta^k(x, p_{k+1}, \ldots, p_N) \) is the price at which bidder 1 will drop out of the auction if the number of bidders who are still active is \( k \), his own signal is \( x \), and the prices at which the other \( N - k \) bidders dropped out were \( p_{k+1} \geq p_{k+2} \geq \ldots \geq p_N \).

   Describe a symmetric Bayes-Nash equilibrium of the open English auction and show that this strategy profile constitutes indeed an equilibrium.

   Is it an equilibrium in dominant strategies? Is it an ex-post equilibrium?

   (c) Show that the symmetric bidding strategies \( \beta(x) = \frac{1}{v(x)} \int_0^x v(y, y) dG(y) \) form a Bayes-Nash equilibrium of the first-price auction.

   (d) Suppose \( N = 2 \), bidder \( i \)'s valuation is \( v_i(\theta_i, \theta_j) = \eta \theta_i + (1 - \eta) \theta_j \). For which \( \eta \) is the outcome of the second-price auction efficient?

2. Consider the Akerlof framework from the lecture.

   **Definition 1.** A competitive equilibrium in the Akerlof model is a price \( p^* \) and a set \( \Theta^* \) of seller types who trade such that

   \[
   \Theta^* = \{ \theta | R(\theta) \leq p^* \} \quad \text{and} \quad p^* = \mathbb{E}[\theta | \theta \in \Theta^*].
   \]

   Suppose the distribution function of \( \theta \) is

   \[
   F(\theta) = \begin{cases} 
   0 & \text{for } \theta < 1 \\
   \theta - 1 & \text{for } 1 \leq \theta \leq 2 \\
   1 & \text{for } 1 < \theta 
   \end{cases}
   \]

   and \( R(\theta) = 0.9 \cdot \theta \).

   Describe a competitive equilibrium of this model.