Microeconomics

1. Uncertainty

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Lotteries

A decision maker faces a choice among a number of risky alternatives.

Each alternative can lead to one of a number of possible *outcomes*.

C is the **finite** set of all possible outcomes with $|C| = N$.

Definition: A *simple lottery* $L = (p_1, \ldots, p_N)$, $p_i \geq 0, \sum p_i = 1$ is a collection of probabilities for the sure outcomes $x_1, \ldots, x_N$. (We write a lottery as a set $\{x_i : p_i\}_{i=1}^{N}$ and denote by $\mathfrak{L}$ the set of all simple lotteries over the set of outcomes C.)

Lotteries

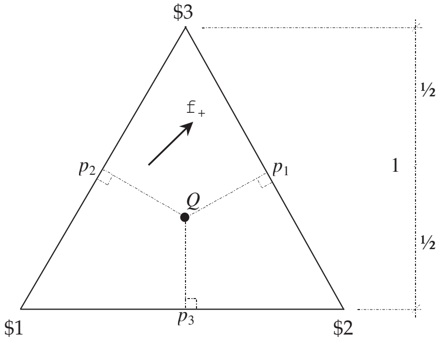
A *simple lottery* can be represented as a point in *simplex*.

Definition: The set $\Delta = \{\mathbf{p} \in {\mathbb{R}_+}^N : \sum p_i = 1\}$ is called a N-dimensional simplex.

In its three outcome (=dimensional) case, a simplex can be graphically represented by an equilateral triangle with altitude 1. Each perpendicular then can be interpreted as the probability of the outcome at the opposing vertex. Thus every point in the triangle represents a lottery.

Lottery

The Lottery $Q = \{1 : p_1, 2 : p_2, 3 : p_3\}$ with all probabilities equal to one third.



Lottery

Definition: Given K simple lotteries, $L_k = (p_1^k, ..., p_N^k)$, $k = 1, ..., K$ and probabilities $\alpha_k \geq 0$ with $\sum_k \alpha_k = 1$, the *compound lottery* $(L_1, ..., L_K; \alpha_1, ..., \alpha_K)$ is the risky alternative that yields the simple lottery L_k with probability α_k for $k = 1, ..., K$.

For any compound lottery, the *reduced* (simple) lottery over final outcomes can be specified

$$p_n = \alpha_1 p_n^1 + ... + \alpha_K p_n^K.$$

or, alternatively,

$$L = \alpha_1 L_1 + \ldots + \alpha_K L_K.$$

Preferences over Lotteries

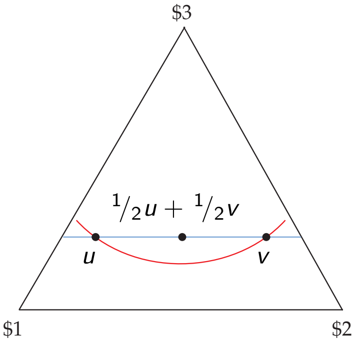
We assume that the DM has a rational (complete and transitive) relation on $\mathfrak{L}$.

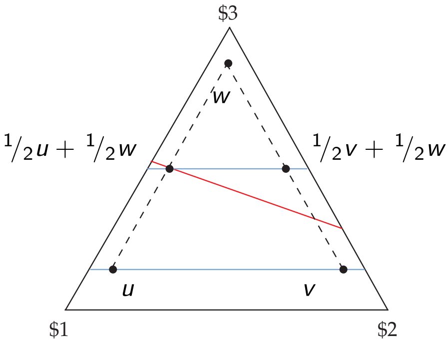
In addition assume:

- *Reduction axiom.* Two compound lotteries are equivalent if they yield the same simple lottery.
- *Continuity.* Suppose that L, L' and L'' are three simple lotteries such that $L \succ L' \succ L''$. Then there exists $\alpha, \beta \in (0,1)$ such that $\alpha L + (1-\alpha)L'' \succ L' \succ \beta L + (1-\beta)L''$
- *Independence.* Suppose that L, L' and L'' are three simple lotteries such that $L \succsim L'$. Then, for any $\alpha \in (0,1)$, $\alpha L + (1-\alpha)L'' \succsim \alpha L' + (1-\alpha)L''$.

Preferences over Lotteries

These assumptions induce indifference curves in the simplex representation to take the form of straight & parallel lines.





Expected utility example

2 alternatives: **A** and **B**

	Bermuda	-500	0
A	0.3	0.4	0.3
B	0.2	0.7	0.1

What we would like to be able to do is to express the utility for these two alternatives in terms of the utility the DM assigns to each individual outcome and the probability that they occur.

$$U(A) = 0.3u_b + 0.4u_{-500} + 0.3u_0.$$

$$U(B) = 0.2u_b + 0.7u_{-500} + 0.1u_0.$$

Expected utility

If at some alternative, the probabilities of the trip to the Bermudas, paying 500 and paying 0 are p_B, p_{-500} and p_0, respectively, then the utility of the alternative is

$$p_B u_b + p_{-500} u_{-500} + p_0 u_0.$$

This representation is called the **expected utility form**.

Expected utility

Definition. The utility function $U : \mathfrak{L} \longrightarrow R$ has an *expected utility form* if there is an assignment of numbers $(u_1, ..., u_N)$ to the N outcomes such that, for every simple lottery $L = (p_1, ..., p_N) \in \mathfrak{L}$, we have

$$U(L) = \sum_i p_i u_i.$$

A utility function $U : \mathfrak{L} \longrightarrow R$ with the expected utility form is called a *von Neumann-Morgenstern expected utility function*.

Expected utility

The expected utility form is preserved only by increasing *linear* transformations.

If $U : \mathfrak{L} \longrightarrow R$ is an expected utility function for the preference relation $\succsim$, then $\widetilde{U} : \mathfrak{L} \longrightarrow R$ is another expected utility function for the preference relation $\succsim$ if and only if there are scalars $\alpha > 0$ and β, such that $\widetilde{U}(L) = \alpha U(L) + \beta$ for every $L \in \mathfrak{L}$.

Expected utility theorem

Suppose that the rational preference relation $\succsim$ on the space of lotteries satisfies the reduction axiom, continuity and independence. Then, there exists a function of the expected utility form that represents $\succsim$. That is, we can assign a number u_n to each outcome $n = 1, ..., N$ such that for any two lotteries $L = (p_1, ..., p_N)$ and $L' = (p'_1, ..., p'_N)$ we have

$$L \succsim L' \text{ if and only if } \sum_{n=1}^{N} u_n p_n \geq \sum_{n=1}^{N} u_n p'_n.$$

Allais paradox

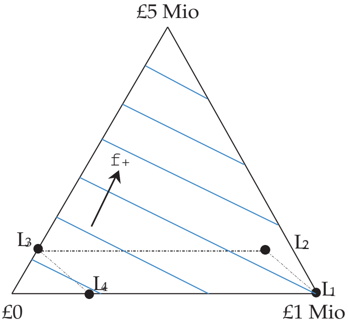
L_1	{$ 1: 1, $ 0: 0}
L_2	{$ 5: 0.1, $ 1: 0.89, $ 0: 0.01 }

L_3	{$ 5: 0.1, $ 0: 0.9}
L_4	{$ 1: 0.11, $ 0: 0.89}

(Amounts are in millions.)

Expected utility

Typically $L_1 \succ L_2$ and $L_3 \succ L_4$—but no parallel, straight level set can represent *both* $L_1 \succ L_2$ *and* $L_3 \succ L_4$!



Hence, the typical choice behaviour described by Allais' paradox *cannot* be represented by expected utility theory.

Money lotteries

Let x be a continuous variable (amounts of money).

We can describe a monetary lottery by means of a cumulative distribution function $F : R \longrightarrow [0, 1]$.

Denote by f the associated density function.

Denote by $u(\cdot)$ the utility function defined on sure amounts of money (Bernoulli utility function) and by $U(\cdot)$, a vN/M utility function.

The expected utility from a lottery $F(\cdot)$ is given by

$$U(F) = \int u(x)dF(x).$$

Risk Aversion

Definition: A DM is called *risk averse* (or said to exhibit *risk aversion*) if, for any lottery $F(\cdot)$, the degenerate lottery that yields the amount $\int xdF(x)$ with certainty is at least as good as the lottery $F(\cdot)$ itself.

If for any $F(\cdot)$ the DM is indifferent between these two lotteries, we say that he is said to be *risk neutral*.

We say that he is *strictly risk averse* if indifference holds only when the two lotteries are the same (when $F(\cdot)$ is degenerate).

Opposite relations refer to *risk loving*.

Risk aversion

Bernoulli utility fns of

- risk averse agents in $[x, u(x)]$ space are concave; the more concave they are, the more risk-averse is the agent
- risk loving agents are strictly convex.

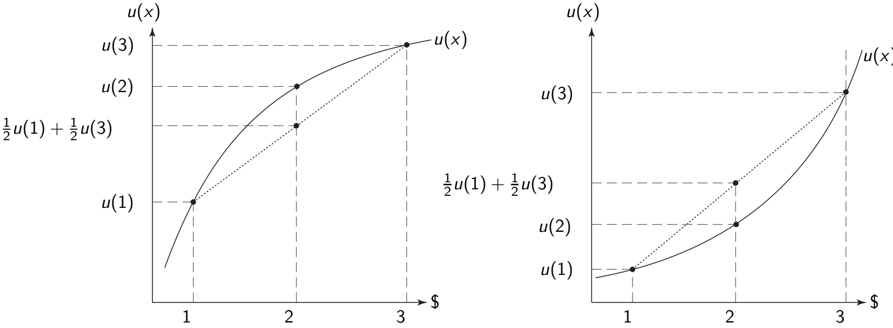
This is a direct consequence of **Jensen's inequality**; for a real continuous concave fn $u(\cdot)$ we have

$$\int_{-\infty}^{+\infty} u(x)dF(x) \leq u\left(\int_{-\infty}^{+\infty} xdF(x)\right).$$

Hence the expected utility is smaller than the utility of an expected value.

Risk aversion

Bernoulli utility functions representing risk averse and risk loving attitudes.



Since the <u>shape</u> of $u(\cdot)$ matters, expected utility functions can be defined up to separate increasing linear transformations.

Risk aversion

A useful concept for the analysis of risk aversion:

Definition: Given a Bernoulli utility function $u(\cdot)$, the *certainty equivalent* of $F(\cdot)$, denoted $c(F, u)$ is the amount of money for which the individual is indifferent between the gamble $F(\cdot)$ and the certain amount $c(F, u)$; that is,

$$u(c(F, u)) = \int_{-\infty}^{+\infty} u(x) dF(x).$$

The DM is risk averse if and only if

$$c(F, u) \leq \int x dF(x) \text{ for all } F(\cdot).$$

Insurance

- strictly risk averse DM has an initial wealth of w
- he runs a risk of a loss of D dollars
- the probability of the loss is π
- one unit of insurance costs q dollars and pays 1 dollar if the loss occurs.

The expected utility of the DM if he acquires α units of insurance

$$(1-\pi)u(w-\alpha q)+\pi u(w-\alpha q-D+\alpha).$$

The DM has to choose how much insurance to acquire (α).

Insurance

If α^* is an optimum, it must satisfy the first order condition

$$-q(1-\pi)u'(w-\alpha^* q)+\pi(1-q)u'(w-\alpha^* q-D+\alpha^*)\leq 0.$$

If the insurance is actuarially fair ($q=\pi$), the FOC requires that

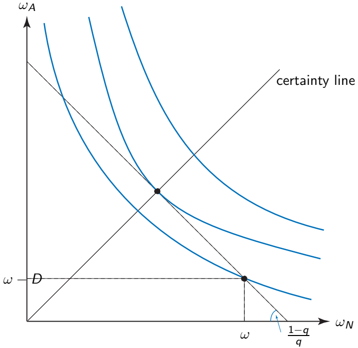
$$u'(w-\alpha^*\pi-D+\alpha^*)-u'(w-\alpha^* q)\leq 0.$$

Since $\alpha^*>0$,

$$u'(w-\alpha^*\pi-D+\alpha^*)=u'(w-\alpha^* q).$$

$u''<0$, therefore, $w-D+\alpha^*(1-\pi)=w-\alpha^* q$ or $\alpha=D$.

Insurance



The slopes of the indifference curves are

$$\frac{d\omega_A}{d\omega_N}\Big|_{dEu=0} = -\frac{(1-\Pi)u'(\omega_N)}{\Pi u'(\omega_N)}.$$

How much insurance will the DM acquire?

$$\frac{(1-\Pi)u'(\omega-\alpha q)}{\Pi u'(\omega-\alpha q-D+\alpha)} = \frac{1-q}{q}.$$

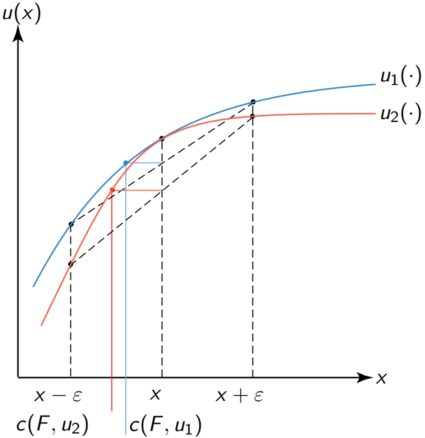
Measurement of risk aversion

Definition: Assume $u(\cdot)$ is C^2 and increasing. Then the Arrow-Pratt *coefficient of absolute risk aversion* $r(\cdot)$ of a Bernoulli utility function defined over outcomes x_i is

$$r(x) = -\frac{u''(x)}{u'(x)}.$$

The following figure should make this idea clearer: at x, both $u_1(\cdot)$ and $u_2(\cdot)$ give the same utility $u(x)$, but the lotteries $\{x - \varepsilon, x + \varepsilon\}$ are evaluated differently by the two agents. Hence the shape of the utility function gives an indication of the degree of risk aversion.

Measurement of risk aversion



Since $c(F, u)$ is bigger for the utility fn with the higher curvature, risk aversion increases with curvature.

Comparisons across individuals

Assume two Bernoulli utility functions $u_1(\cdot)$ and $u_2(\cdot)$. When can we say that $u_2(\cdot)$ is more risk averse than $u_1(\cdot)$?

The following statements are equivalent

1. $r(x, u_2) \geq r(x, u_2)$ for every x
2. there exists an increasing concave function $\psi : R \longrightarrow R$ such that $u_2(x) = \psi(u_1(x))$ for all x
3. $c(F, u_2) \leq c(F, u_1)$ fore any $F(\cdot)$.

Comparisons across wealth levels

Definition: The Bernoulli utility function $u(\cdot)$ exhibits *decreasing absolute risk aversion* (DARA) if $r(x, u)$ is a decreasing function of x.

Definition: The Bernoulli utility function $u(\cdot)$ exhibits *constant absolute risk aversion* (CARA) if $r(x, u)$ is a constant function of x. For example if $u(x) = -e^{-\alpha x}$ for $\alpha > 0$, then $r(x, u) = \alpha$.

Definition: The Bernoulli utility function $u(\cdot)$ exhibits *increasing absolute risk aversion* (IARA) if $r(x, u)$ is an increasing function of x.

Stochastic dominance

We are interested in knowing whether one distribution offers higher returns then another.

2 possible criteria:

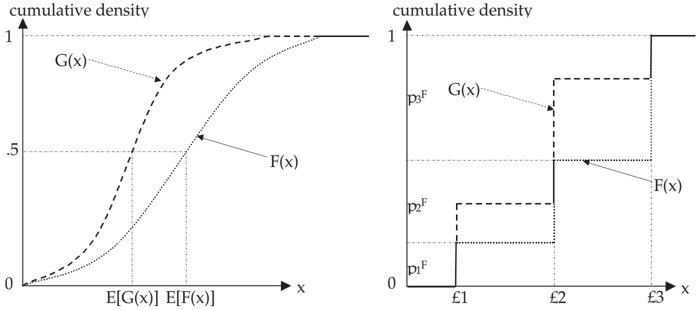
1. Every DM prefers $F(\cdot)$ to $G(\cdot)$.

2. For every amount of money x, the probability of getting at least x is higher under $F(\cdot)$ than under $G(\cdot)$.

Stochastic dominance

Definition The probability distribution of F *first-order stochastically dominates* (fosd) that of G if for non-decreasing $u(\cdot)$

$$\int_{-\infty}^{+\infty} u(x)dF(x) \geq \int_{-\infty}^{+\infty} u(x)dG(x).$$



Stochastic dominance

Definition

Let two probability distributions F, G share the same mean (i.e. $\int x dF(x) = \int x dG(x)$). Then distribution F *second-order stochastically dominates* (sosd) distribution G if, for concave $u(\cdot)$

$$\int u(x)dF(x) \geq \int u(x)dG(x).$$

Sosd is about distributions with the same expectation but different risks. An example is a mean preserving spread of a given distribution. If F sosd's G, for all x_0 we have

$$\int_{-\infty}^{x_0} G(t)dt \geq \int_{-\infty}^{x_0} F(t)dt.$$

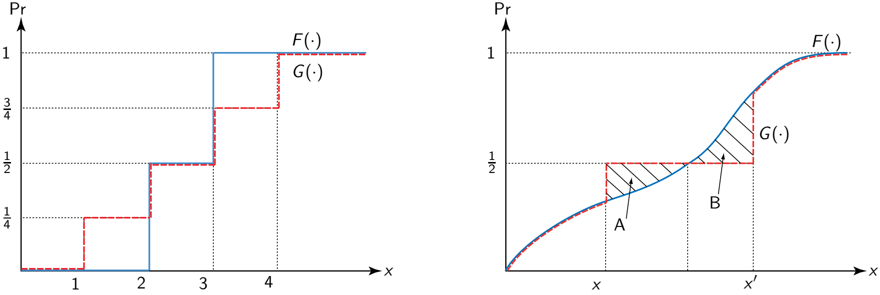
Stochastic dominance

The idea of Sosd is captured by a **mean-preserving spread** of a given distribution $F(\cdot)$.

Example: Let

- $F(\cdot)$ be an even probability dist'n between 2\$ and 3\$
- now 'spread' the 2\$ outcome to an even probability between 1 and 2\$, and the 3\$ outcome to an even probability between 3 and 4\$ (i.e. giving probability $\frac{1}{4}$ to all outcomes)
- call the resulting distribution $G(\cdot)$
- notice that our probability manipulations do not change the mean of the distributions $\mathbb{E}[F(\cdot)] = \mathbb{E}[G(\cdot)]$

Stochastic dominance



Sosd and mean-preserving spreads of some dist'n $F(\cdot)$.

Saying that $G(\cdot)$ is a mean-preserving spread of $F(\cdot)$ is equivalent to saying that $F(\cdot)$ second order stochastically dominates $G(\cdot)$.