Abstract: We show that only trivial choice rules are ex-post implementable in multi-dimensional mechanism design settings with generic, interdependent valuations. That is, ex-post implementation implies that the same alternative must be chosen irrespective of agents’ signals. The proof is based on the observation that implementation of a non-trivial choice rule is only possible if a geometric condition combining the gradients of the agents’ valuation functions is satisfied. This condition amounts to a system of equations that has no solution generically. As a consequence, other notions of implementation must be considered in frameworks with interdependent valuations. We illustrate the contrast between ex-post and Bayes-Nash implementation in a two-dimensional auction set-up.
1 Introduction

A growing literature (in particular in the sub-field of auction theory) adapts the Vickrey-Clarke-Groves idea to the case of interdependent valuations, i.e., where an agent’s valuation may depend also on information available to other agents. Since dominant strategy implementation is usually impossible in this framework, this literature has used the weaker concept of ex-post implementation: even after learning the signals of others, agents should not be willing to deviate from their equilibrium strategies. This notion is very interesting because agents need not know the distribution of others’ signals in order to play3.

The Vickrey-Clarke-Groves mechanisms (see Vickrey (1961), Clarke (1971), Groves (1973)) align the interests of several heterogenous agents by assigning to each agent a transfer equal to the sum of valuations of the other agents in the chosen social alternative. With such transfers, all individual payoff maximization problems coincide with the maximization of social surplus, yielding the well-known dominant strategy implementability of the efficient choice rule. Besides the assumption of quasi-linear utility, the basic VCG argument heavily depends on the assumption of ”private values”, e.g., an agent’s valuation function depends only on private information available to that agent, and on the social alternative.


In order to obtain positive results, all above papers had to assume that agents’ signals are one-dimensional, and that valuations satisfy a Single Crossing Property (SCP). This property can be stated as follows: For every agent $i$, there is an order on the alternatives such that a higher signal $s^i$ makes higher alternatives more preferable than lower alternatives, both for agent $i$ and for the entire society.4. In an auction setting this translates to

3See Bergemann and Morris (2002) for a formal treatment of this issue, which is also connected to Wilson’s concern about detail-free mechanisms (but, see McLean-Postlewaite,2002 for a challenging view on this point). See Chung and Ely (2003) and Meyer-ter-Vehn and Moldovanu (2003) for characterizations of ex-post implementation in various frameworks.

4This condition is different from the single crossing property used in the signaling literature. In that context, SCP is solely a condition on one agent’s preferences and has
the condition that \( i \)'s information is more important to her than to any other agent.

The definition of single crossing necessitates an order on each agent's signal space, and such an assumption does not generalize well to multi-dimensional type spaces. Maskin (1992) gave an early two-dimensional example where efficiency is not achievable in ex-post equilibrium. In a more general setting, JM (2001) show that even the weaker requirement of efficient Bayes-Nash implementation imposes non-generic conditions on the parameters of their model whenever the signal space of at least one agent is multi-dimensional.

Given the generic impossibility of efficient implementation in the multi-dimensional case, a natural question is: What other choice rules can be implemented?

In this paper we prove a strong negative result pertaining to the above question: Generically, only trivial choice rules - for which the same alternative is chosen irrespective of agents' signals - are implementable. In other words, any given mechanism has either only ex-post equilibria that involve complete pooling, or no ex-post equilibria at all. The main step of the proof derives a geometric condition on valuation functions that must be satisfied for a non-trivial choice rule to be implementable. The impossibility result follows by noting that this condition, which requires the gradients of agents' payoff functions to point in the same direction, is generically impossible to satisfy\(^5\). It is important to note that, unlike the impossibility result on efficient implementation in JM (2001), this result crucially relies on the fact that multiple agents hold private information\(^6\).

Our main result can be seen as a criticism on, or a limitation of the otherwise appealing concept of ex-post implementation. In particular, it means that, with interdependent values, our attention must be re-directed toward Bayes-Nash implementation. In order to illustrate the contrast between ex-post and Bayes-Nash implementation, we compute a Bayes-Nash equilibrium of a second-price auction with multi-dimensional signals and multiplicative valuations - this is, to our knowledge, the first such example in the literature.

The rest of the paper is organized as follows: In Section 2 we describe the mechanism design problem, define the equilibrium concept, and derive some

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\(^5\)To be precise: The condition is not satisfied by a residual set of valuation functions.  
\(^6\)In a framework with a unique informed agent, one can set prices (possibly contingent on the designer' signals) for each alternative, and let the informed agent choose the alternative he likes best (given the prices). Obviously this leads to a non-trivial, ex-post implementable choice rule. Our main result is due to the impossibility of simultaneously satisfying the incentive constraints of several agents.
helpful implications. In Section 3 we prove the generic impossibility result about implementation in ex-post equilibria. Section 4 contains a discussion and several remarks about the main result. Section 5 concludes by redirecting attention to weaker implementation concepts. Appendices A and B contain several proofs that would interrupt the flow of the argument in the main text. Appendix C, by William Zame, contains a general result about polynomial differential operators applied to pairs of functions. A corollary of this result is used in the main text to show that the geometric condition implied by ex-post implementability cannot be satisfied generically.

2 The Model

Consider a setting with \( N \in \mathbb{N} \) agents, \( i, j \in \mathcal{N} \), who are affected by a decision among \( K \in \mathbb{N} \) alternatives, \( k \in \mathcal{K} \). Agent \( i \)'s utility, \( u^i = v^i_k - t^i \), takes into account the chosen alternative \( k \), and a monetary payment \( t^i \in \mathbb{R} \). Her valuation \( v^i_k = v^i_k(s) \) for alternative \( k \) depends on the state of the world \( s \in S \). We denote by \( v^i = (v^i_k) \) the vector of \( i \)'s valuations of alternatives, by \( v^N = \sum_{i \in \mathcal{N}} v^i_k \) the social surplus in alternative \( k \), and by \( v^N = (v^N_k) \) the vector of surplus functions.

Each agent holds private information \( s' \in S' \) on the state of the world \( s \in S \). The signal \( s' \) results from an exogenous draw. There is no loss of generality in assuming that the agents' joint information \( (s')_{i \in \mathcal{N}} \) completely determines the state of the world \( s \). We thus identify states of the world with signal combinations: \( S = \prod_{j \in \mathcal{N}} S^j \). We adopt the usual notation \( s^{-i} = (s')_{j \in \mathcal{N}, j \neq i} \in S^{-i} \) and \( s = (s'; s^{-i}) \) when we focus on agent \( i \). We assume\(^7\) that \( S^i = [0, 1]^{d^i} \), and that \( v \) admits a continuously differentiable extension to an open neighborhood of \( S \). We denote by \( \nabla_s \), the \( d^i \)-dimensional vector of derivatives with respect to \( s^i \), and by \( \partial_{\rho} \), the directional derivative in direction \( \rho \in \mathbb{R}^{d^i} \).

We are interested in choice rules \( \psi : S \rightarrow K \) with the property that there are transfers functions \( t^i : S \rightarrow \mathbb{R} \) such that truth-telling is an ex-post equilibrium in the incomplete information game resulting from the direct

\(^7\)A more general model would distinguish between joint information \( s = (s')_{i \in \mathcal{N}} \) and states of the world \( \omega \in \Omega \), on which agents’ valuations would depend \( v^i_k = w^i_k(\omega) \). In such a model each signal combination \( s \) induces a probability distribution \( p_s \) over \( \Omega \). Our model is a reduced form of this by setting \( v^i_k(s) := \int_{\Omega} w^i_k(\omega) \, dp_s(\omega) \). As \( i \)'s signal \( s' \) is exogenously drawn, this imposes no loss of flexibility in the model.

\(^8\)This assumption is purely for the sake of concreteness. Assuming \( S^i \) to be the closure of any open connected subset of \( \mathbb{R}^{d^i} \) would suffice as well.
revelation mechanism \( (\psi, (t^i)_{i \in \mathcal{N}}) \), i.e.

\[
v^i_{\psi(s)}(s) - t^i(s) \geq v^i_{\psi(s^i,s^{-i})}(s) - t^i(s, s^{-i})
\]

for all \( s^i, \tilde{s}^i \in S^i \) and \( s^{-i} \in S^{-i} \), where \( s := (s^i, s^{-i}) \). We shall call such \( \psi \) implementable. We call a choice rule \( \psi \) trivial, if it is constant on the interior \( \mathring{S} \) of the type space.

2.1 The Taxation Principle

The taxation principle is well-known from the monopolistic screening literature: instead of asking an individual for the private information needed to decide on an alternative and a transfer, the central authority can, equivalently, post prices for the various alternatives, and let the individual choose among them. Monopolistic screening models can be seen as mechanism design with one agent, and an ex-post equilibrium can be seen as an aggregation of one-agent problems where, in equilibrium, all agents necessarily agree on the best alternative. It is therefore not surprising that there is a taxation principle for multi-agent ex-post mechanism design.

**Lemma 2.1 (Taxation Principle)** A choice rule \( \psi \) is implementable, if and only if for all \( i \in \mathcal{N}, k \in \mathcal{K} \) and \( s^{-i} \in S^{-i} \), there are transfers \( t^i_k(s^{-i}) \) such that:

\[
\psi(s) \in \arg \max \left\{ v^i_k(s) - t^i_k(s^{-i}) \right\}.
\]

**Proof.** See Appendix A, or Chung and Ely (2003).

Note that the personalized prices \( t^i_k(s^{-i}) \) charged to agent \( i \) may depend on other agents’ signals \( s^{-i} \).

2.2 Monotonicity

Implementable choice rules \( \psi \) satisfy a monotonicity property that can be interpreted as an alignment of \( i \)'s preferences with the preferences implicit in the choice rule \( \psi \): If the change in \( i \)'s signal from \( s^i \) to \( s'^i \) makes alternative \( k \) relatively more preferable for agent \( i \) than alternative \( l \), then \( \psi \) must respect these preferences by not choosing \( k \) at signal \( s^i \) and \( l \) at signal \( s'^i \).

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9 Note that the revelation principle applies to ex-post equilibrium.

10 Restricting attention to the interior of the type space is justified since the interior has full measure. This assumption is necessary since the main geometric arguments in the proof fail on the boundary of the type space. Alternatively, we could have assumed open type spaces to start with.

11 To be precise: transfers \( \left( \{ t^i_k(s^{-i}) \} \right)_{k \in (\mathbb{R} \cup \{ \infty \})} \setminus (\infty, ..., \infty) \).
Lemma 2.2 An implementable choice rule \(\psi\) is monotonic in the following sense: For every agent \(i\) and for all signals \(s = (s^i; s^{-i}), s' = (s'^i; s'^{-i}) \in S\), such that
\[
v^i_k(s') - v^i_k(s) > v^i_l(s') - v^i_l(s),
\]
\(\psi(s) = k\) implies that \(\psi(s') \neq l\).

Proof. See Appendix A.

The above property is essentially the Positive Association of Differences (PAD) condition in Roberts (1979) where it was stated in a private values framework.

3 The Impossibility Theorem

In this section we derive our main result, Theorem 3.3. We restrict ourselves to two agents \(i, j \in \{1, 2\}\) and two possible alternatives \(\{k, l\}\). The impossibility result immediately generalizes as this ”2 by 2” model is naturally embedded in every model with more agents and alternatives. As agents’ incentives are only responsive to differences in payoffs, we now introduce relative valuations \(\mu^i\) and relative transfers \(\tau^i\):
\[
\mu^i(s) = v^i_k(s) - v^i_l(s), \text{ and }
\tau^i(s^{-i}) = t^i_k(s^{-i}) - t^i_l(s^{-i}).
\]

For technical reasons, we assume that relative valuations satisfy the mild requirement \(\nabla_{s^i} \mu^i \neq 0\), where ”\(\nabla_{s^i}\)” denotes the \(d^i\)-dimensional vector of partial derivatives with respect to \(i\)'s signal \(s^i\).

Before we embark on the proof of the general result, we illustrate it with an example.

Example 3.1 Consider the following single-object allocation problem:

- There are two bidders \(i \in \{1, 2\}\);
- Bidders have two-dimensional signals \(s^i = (p^i, c^i) \in [0, 1]^2\);
- The valuation of bidder \(i\) for the unit is given by \(v^i(s^i, s^{-i}) = p^i + c^i c^{-i}\).

We claim that the second-price auction has no ex-post equilibrium in symmetric bidding strategies. Any (non-symmetric) equilibrium in this auction has one bidder bidding prohibitively high, i.e. \(b^i(p^i, c^i) \geq 1 + c^i\), while the other bidder bids zero\(^{12}\). In such an equilibrium, bidder \(i\) always wins the object, and the implemented choice rule is trivial.

\(^{12}\)Both these strategies are weakly dominated.
Proof. In an ex-post equilibrium, bidder $i$ must be indifferent about winning or losing for all signal profiles $(s^i, s^{-i})$ such that $b^i(s^i) = b^{-i}(s^{-i})$. In a symmetric ex post equilibrium where $b^i(p, c) = b^j(p, c) = b(p, c)$, this implies $b(p, c) = p + c^2$. With these strategies, a bidder with type $(0, 1)$ bids 1 and wins against an opponent with type $(0.5, 0)$ and bid 0.5. But, in this case, the winner pays 0.5 for the object, and regrets her bid. Thus, the only symmetric candidate strategy is not an ex-post equilibrium. The second part of the claim (about asymmetric equilibria) follows a similar line of argument and is omitted here. ■

It is tempting to put the blame for the non-existence of a symmetric ex-post equilibrium on the fact that a one-dimensional bid is insufficient to represent an agent’s two-dimensional information. Proposition 3.6 will prove that this argument is faulty: we show there that no mechanism whatsoever (i.e., not even a mechanism that allows agents to fully express their two-dimensional signals) can implement a non-trivial choice rule. Any mechanism will either have no ex-post equilibria at all, or only ex-post equilibria that involve complete pooling of the agents’ types, as is the case in the example above. In Section 5 we construct a non-trivial symmetric Bayes-Nash equilibrium for the second price auction, and show how it fails the condition of ex-post implementation.

Before stating the main result, we introduce the precise topological notion of genericity used below.

Definition 3.2 Let $C^1(S, \mathbb{R}^2)$ be the (Banach) space of maps $S \to \mathbb{R}^2$ that admit continuously differentiable extensions to an open neighborhood of $S$, equipped with the topology of uniform convergence of maps and first derivatives. Consider the open subset $Z := \{ (\mu^1, \mu^2) \in C^1(S, \mathbb{R}^2) : \nabla_x \mu^i \neq 0 \}$ equipped with the $C^1$-topology. We say that some property is satisfied for generic relative valuation functions if it is satisfied for all functions $\mu$ in a residual subset$^{13}$ $V \subseteq Z$.

Since $Z$ is a Baire-space, residual subsets are dense and should thus be thought of as large.

Theorem 3.3 Assume that signal spaces are multi-dimensional. For generic relative valuations $\mu \in Z$, only trivial choice rules are implementable.

$^{13}$A subset $X$ of a topological space $Y$ is residual, if it is the countable intersection $X = \bigcap_{\nu \in \mathbb{N}} X_{\nu}$ of open and dense sets $X_{\nu} \subseteq Y$. Note that this topological notion of "largeness" need not coincide with the measure-theoretic notion of "almost everywhere" when $Y$ is equipped with both a topology and a measure, e.g. $Y = [0, 1]$.  

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The proof of the above Theorem consists of two major steps. Proposition 3.6 below shows that the existence of a non-trivial ex-post implementable choice rule implies a geometric condition on the gradients of the relative valuation functions. Corollary 2 in Appendix C, due to William Zame, shows that the conditions derived in Proposition 3.6 cannot be satisfied for generic valuation functions.

The proof of Proposition 3.6 relies on a geometric argument on the boundary that separates the areas (in the signal space $S$) where alternatives $k$ and $l$, respectively, are chosen.

**Definition 3.4** Consider a choice rule $\psi$. The **indifference set** $I$ is defined by: $I := \psi^{-1}(\{k\}) \cap \psi^{-1}(\{l\}) \cap \overset{.}{S}$. For an **indifference signal** $\hat{s} \in I$, we define the indifference set with fixed $\hat{s}^i$ to be $I^i(\hat{s}) := \{s \in I : s^i = \hat{s}^i\}$.

By the connectivity of $\overset{.}{S}$, a choice rule is trivial if and only if its indifference set $I$ is empty. Even though our result does not require relative transfers to be continuous, it is worth noting that, for mechanisms with continuous relative transfers, $I$ coincides with the set of agents’ indifference signals\textsuperscript{14}, thus justifying the term “indifference set”.

**Lemma 3.5** Let $(\psi, t)$ be an ex-post incentive compatible mechanism with continuous relative transfers.

1. For every $\hat{s} \in \overset{.}{S}$ and $i \in \{1, 2\}$, we have
   $$\mu^i(\hat{s}) - \tau^i(\hat{s}^{-i}) = 0 \iff \hat{s} \in I$$

2. For all $\hat{s} \in I$, $I^i(\hat{s})$ is a $(d^i - 1)$-dimensional submanifold of $\overset{.}{S}$.

**Proof.** See Appendix A. ■

We say that two vectors $x, y \in \mathbb{R}^d$ are co-directional if there is $\lambda \geq 0$ with $\lambda x = y$.

**Proposition 3.6** Let $(\psi, t)$ be a non-trivial ex-post incentive compatible mechanism. Then one of the following conditions must be satisfied:

\textsuperscript{14}Moreover, if the choice rule $\psi$ maximizes some objective function $(P_k, P_l)$ (e.g. social surplus $\psi(s) \in \arg\max_{s \in \overset{.}{S}} \{v^N_k(s), v^N_l(s)\}$ with $\nabla (P_k - P_l) \neq 0$, the set of interior signals where the planner is indifferent between the alternatives is also $I$.}
1. Assume that the relative transfers $\tau^i$ are continuous on $S^j$ for all $i \in \{1, 2\}$. Then, there is an indifference signal $\hat{s} \in I$, and a vector $y \in \mathbb{R}^d$, such that the relative valuations of the agents satisfy:

$$\forall s \in I^i(\hat{s}), \nabla_s \mu^i(s) \text{ and } (\nabla_s \mu^j(s) - y) \text{ are co-directional.} \quad (3)$$

2. Assume that relative transfers $\tau^j$ are discontinuous at $b^i s \in S^i$ for some $j \in \{1, 2\}$. Then, locally, agent $i$’s incentives do not depend on $s^j$. That is, there is a vector $y \in \mathbb{R}^d$, such that

$$\forall s \in (\hat{s}^i, \hat{s}^j), \nabla_s \mu^i(s) \text{ and } y \text{ are co-directional.} \quad (4)$$

where $S^j \subset S^i$ is open and contains a non-empty, $(d^j - 1)$-dimensional manifold $\{s = (\hat{s}^i, s^{-i}) : \mu^{-i}(s) = c\}$ for some $c \in \mathbb{R}$.

Let us sketch the proof of this proposition in a simple case: assume that relative transfers in the incentive compatible mechanism $(\psi, t)$ are differentiable (and hence continuous), and set $y := \nabla_s \tau^j(\hat{s}^i)$. Both agents are indifferent between the alternatives at any $s = (\hat{s}^i, s^j) \in I^i(\hat{s})$. Thus, according to Lemma 3.5, we must have:

$$\mu^i(s) - \tau^i(s^j) = 0$$
$$\mu^j(s) - \tau^j(s^i) = 0$$

Consider now slight variations $s^i$ in a neighborhood of $\hat{s}^i$: if the agents’ valuations were not aligned in the sense of condition (3), there would be a signal $s^h$ close to $\hat{s}^i$, such that agent $i$ strictly prefers alternative $k$ at signal $(s^h, s^j)$, and agent $j$ strictly prefers $l$. But this would contradict the taxation principle.

Condition (3) (and similarly (4)) is a strong condition on relative valuations $\mu$. For fixed $s \in I^i(\hat{s})$, it is equivalent to a system of $d^i - 1$ equations. This system has to be satisfied for all $s \in I^i(\hat{s})$ with respect to the same parameters $y$. As $I^i(\hat{s})$ is a $(d^i - 1)$-dimensional submanifold of $S$ (see Lemma 3.5), this amounts to "$(d^i - 1)$ continua" of equations on $\mu$, with only $d^i$ free parameters (the coordinates of $y$). It is intuitive that, generically, such a system cannot have a solution. For a further simple illustration see also the proof of Corollary 4.1 below which treats a setting with linear valuation functions.

We will now formally establish why conditions (3) or (4) cannot be satisfied generically. The second main step in the proof of our main result is as follows:
For two vectors \( x, y \in \mathbb{R}^d \), define the cross product\(^{15} \) \( x \times y \) to be the vector in \( \mathbb{R}^{d(d-1)/2} \) obtained by listing all terms of the form \( x_h y_l - y_h x_l \), \( h = 1, 2, \ldots, d - 1 \), \( l = h + 1, \ldots, d \). It is obvious that \( x \times y = 0 \) if \( x \) and \( y \) are co-directional. Thus, if some given relative valuation functions \( (\nabla s_i) \) implementable.

Thus, if some given relative valuation functions \( (\mu^i, \mu^j) \) allow for non-trivial ex-post implementation (with continuous or discontinuous transfers), we obtain by Proposition 3.6 that \( \nabla s_i \mu^i (s) \times (\nabla s_j \mu^j (s) - y) = 0 \) for all \( s \in I^i (\hat{s}) \). This shows that a first-order polynomial differential equation (with constant coefficients) in the variables \( (\mu^i, \mu^j) \) is satisfied on a level set\(^{16} \). But Corollary 2 in Appendix C shows that generic pairs of differentiable functions do not have this property. Hence, ex-post implementability is generically impossible. This concludes the proof of Theorem 3.3.

Given valuation functions \( v \), it can be checked directly if the conditions in Proposition 3.6 are satisfied. Here is an example:

**Example 3.7 (3.1 resumed)** In the single-unit allocation problem among two bidders where \( v^i (p, c) = p^i + c^i c^{-i} \), only trivial allocation rules are ex-post implementable.

**Proof.** The relative valuations\(^{17} \) are given by: \( \mu^i = p^i + c^i c^j \) and \( \mu^j = -p^j - c^i c^j \). Assume that \( (\psi, t) \) is a non-trivial ex-post incentive compatible mechanism with continuous relative transfers. Condition (3) requires the existence of an indifference signal \( \hat{s} \in (0, 1)^4 \), of a vector \( (A, B)^T \), and of a function \( \lambda (c^j) \in \mathbb{R}^+ \) such that:

\[
\lambda (c^j) \begin{pmatrix} 1 \\ c^j \end{pmatrix} = \begin{pmatrix} 0 - A \\ -c^j - B \end{pmatrix}
\]

for all \( c^j \) in a neighborhood of \( \partial \hat{s} \). By the first equation, \( \lambda (c^j) \) is independent of \( c^j \) and equal to \( -A \). Note, however, that the second equation \( \lambda (c^j) c^j = -c^j - B \) can be satisfied for a continuum of \( c^j \) only if \( \lambda (c^j) \equiv -1 \). This is in contradiction to \( \lambda (c^j) \in \mathbb{R}^+ \). Alternatively, a consideration of the cross product \( -c^j - B + A c^j = 0 \) yields \( A = 0 \) and \( B = 1 \). This shows again that \( \nabla s_i \mu^i (s) \) and \( (\nabla s_j \mu^j (s) - (1, 0)^T) \) are co-linear but not co-directional.

To see that condition (4) isn’t satisfied either, note that the direction of \( \nabla s_i \mu^i (s) = (1, c^j)^T \) cannot be locally independent of \( s^j \). Thus, non-trivial implementation is not possible with discontinuous transfers either. ■

\(^{15}\)This is a simple generalization (suited for our purpose) of the usual cross product in \( \mathbb{R}^3 \).

\(^{16}\)A set \( I = I^i (\hat{s}) \subset \mathcal{S} \) is a level set, if \( I^i (\hat{s}) = \left\{ s = (\hat{s}^i, s^{-i}) \in \hat{s} : v^j (s) = c \right\} \) for some \( i \in \{1, 2\}, \hat{s}^i \in \mathcal{S}^i, \) and \( c \in \mathbb{R} \). The sets on which conditions (3) and (4) have to hold, respectively, contain level sets.

\(^{17}\)The two alternatives are: ”i gets the object” and ”j gets the object”.

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Remark 3.8 It is crucial for our argument that both agents’ signals are multi-dimensional. The condition $d_i \geq 2$ requires two multi-dimensional vectors to be parallel (instead of two scalars). The condition $d_j \geq 2$ requires that this is true with respect to the same parameter $y$ on a $(d_j - 1)$-dimensional manifold, i.e. for a continuum of signals $s$.

4 Discussion and Further Remarks

4.1 Dictatorial Choice Rules

It is instructive to compare our result with the Gibbard-Satterthwaite Theorem (see Gibbard (1973), and Satterthwaite (1975)). These authors focused on dominant strategy implementation in a framework with private values where utility functions need not be quasi-linear. Under some assumptions, they showed that any exhaustive\footnote{I.e., any alternative is chosen for some realization of types.} and dominant strategy implementable choice rule must be dictatorial. Their model is, in one respect, more general than the one considered here because it does not impose quasi-linearity on utility functions. On the other hand, it is more restrictive because of the private values assumption\footnote{Note also that Gibbard-Satterthwaite’s result assumes that there are at least three alternatives and that all preference orderings on the set of alternatives are possible (universal preference space). By contrast, our result only requires that there are at least two alternatives, and that at least for two agents, preferences cannot be parameterized by a signal whose dimension is less than two.}. In spite of these differences, the question arises whether dictatorial choice rules are implementable in our model. Let us call a choice rule dictatorial if there is one agent $i$, the dictator, with the property that $\psi$ maximizes $i$’s valuation. In our model it is not feasible to let the dictator choose her most preferred outcome because her preferences depend on others’ information. By Theorem 3.3, this information cannot be truthfully elicited in ex-post equilibrium (because a dicatotrial rule generically induces a non-trivial choice rule). Note that the notion of dictatorship is ambiguous in an interdependent valuations model: we could also call agent $i$ dictator if the choice rule $\psi$ only depends on $s^i$. By Theorem 3.3, these dictatorial rules are not implementable either.

4.2 The Efficient Choice Rule

The important role of signals’ dimensionality has been emphasized by JM (2001) in the context of efficient Bayes-Nash implementation. Although the generic impossibility of ex-post efficient implementation is obviously a
corollary of Theorem 3.3, we give here an independent argument. This will highlight both the similarities and the differences between the two results.

For the sake of illustration, assume here that only one agent $i$ holds private information, and that $\nabla s_i u^N \neq 0$. Efficient ex-post implementation implies that there is a difference in transfers $\Delta = \tau^i$, such that society is indifferent between the alternatives if and only if this is the case for agent $i$. Mathematically, the level sets $(\mu^i)^{-1} (\Delta)$ must coincide with the indifference set of the efficient choice rule $I^{eff} := (\mu^N)^{-1} (0)$. Therefore, we obtain that:

$$\nabla s_i u^i (s) \text{ and } \nabla s_i u^N (s) \text{ are co-directional for all } s \in I^{eff}. \quad (5)$$

Applying condition (5) to the linear valuations in JM (2001) yields their condition of congruence between private and social rates of information substitution. It is evident that this is a non-generic condition on valuations.

The condition for efficient ex-post implementation, equation (5), is stronger than the condition for non-trivial implementation, equation (3), yet structurally similar. The intuition for (5) is that agent $i$’s preferences must be aligned with social preferences in order to ensure efficient implementation. In equation (3) the role of social preferences is played by agent $j$’s preferences. But agent $j$’s preferences can be altered via the endogenous transfer $\tau^j$, whereas the social preferences are fixed by the valuation functions. Therefore, proving the impossibility of non-trivial ex-post implementation is considerably harder than the impossibility of efficient ex-post implementation. While the impossibility of efficient implementation is already obtained with a single agent having a multi-dimensional signal, our present impossibility result requires that at least two agents have multi-dimensional signals (see Remark 3.8 above and Footnote 6 in the Introduction).

### 4.3 Linear Valuations

In order to facilitate a better understanding of the condition for ex-post implementation we now apply Theorem 3.3 to a setting with linear valuations and 2-dimensional signals $s^i = (s^i_k, s^i_l) \in [0,1]^2$. In this case, non-trivial implementation implies a simple algebraic condition on the coefficients of the valuation functions. Define valuations $v$ by:

$$v^i_k (s) = a^i_k s^i_k + b^i_k s^i_k s^j_k = s^i_k (a^i_k + b^i_k s^j_k)$$

$$v^i_l (s) = a^i_l s^i_l + b^i_l s^i_l s^j_l = s^i_l (a^i_l + b^i_l s^j_l)$$

where $a^i_{k,l}, b^i_{k,l} \neq 0$. 

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Corollary 4.1 For given valuations $v$ as above, a non-trivial choice rule $\psi$ is implementable only if

$$b^1_k b^2_\ell - b^1_\ell b^2_k = 0. \quad (6)$$

Proof. If $(\psi, t)$ is a non-trivial incentive compatible ex-post mechanism with continuous relative transfers $\tau^i$, condition (3) must be satisfied: there is $\bar{s} \in I$, $\Delta := \mu^i(\bar{s})$ and $(A, B)^T \in \mathbb{R}^2$, such that for all $s \in I^i(\bar{s})$

$$\left( a^k_s + b^k_s s^j_s \right) \left( -a^l_s - b^l_s s^j_s \right) \text{ and } \left( b^k_s s^j_s - A \right) \left( -b^l_s s^j_s - B \right) \text{ are co-directional}$$

For this to be true at some $s$, the cross product of these vectors must vanish, implying the following condition on the coefficients $a, b$ and the signal $s$:

$$\left( a^k_s + b^k_s s^j_s \right) \left( -a^l_s - b^l_s s^j_s \right) \left( b^k_s s^j_s - A \right) \left( -b^l_s s^j_s - B \right) = 0. \quad (7)$$

We now argue, that the above condition can be satisfied for all $s$ in the set $I^2(\bar{s})$ only if the coefficients $a, b$ satisfy the algebraic condition (6).

The one-dimensional indifference set $I^i(\bar{s})$ can be parametrized by $s^j_s = \frac{\Delta}{a^j_s + b^j_s \bar{s}^j_s} + \frac{a^j_s + b^j_s \bar{s}^j_s}{a^j_s + b^j_s \bar{s}^j_s}$, where $\Delta = \mu^i(\bar{s})$. As $a^k_s, b^k_s \neq 0$, we can assume w.l.o.g. that $a^k_s + b^k_s \bar{s}^j_s, a^j_s + b^j_s \bar{s}^j_s \neq 0$. Substituting for $s^j_s$ in condition (7), we see that this equation can only hold on all of $I^i(\bar{s})$ if the coefficient of the quadratic term in $s^j_s$ vanishes, i.e. if $\frac{a^j_s + b^j_s \bar{s}^j_s}{a^j_s + b^j_s \bar{s}^j_s} \left( -b^l_s s^j_s - b^j_s \right) = 0$. This implies condition (6). Finally, for the case of discontinuous transfers $\tau^i$, condition (4) reduces here to $b^k_s = b^j_s = 0$, so that condition (6) is satisfied. 

4.4 Separable Valuation Functions

Generic impossibility in the sense of Theorem 3.3 leaves space for large classes of interesting (yet non-generic) valuations for which non-trivial ex-post implementation is feasible. For example, this is the case for separable valuation functions, i.e. $v$ for which there are functions $f^i_k : S^i \rightarrow \mathbb{R}$, $h^i_k : S^{-i} \rightarrow \mathbb{R}$ such that:

$$v^i_k (s) = f^i_k (s^i) + h^i_k (s^{-i}).$$

Definition 4.2 A choice rule $\psi$ is an affine maximizer, if there are agent-specific weights $\alpha^i \in \mathbb{R}^+$ and alternative specific weights $\lambda_k \in \mathbb{R}$, such that

$$\psi (s) \in \arg \max_k \left\{ \sum_i \alpha^i f^i_k (s^i) + \lambda_k \right\}.$$
Meyer-ter-Vehn and Moldovanu (2003) show that, with separable valuations, affine maximizers are ex-post implementable. Somewhat more surprisingly, under some technical assumptions, they show that affine maximizers are the only implementable choice rules. This characterization result relies on a result by Roberts (1979), who considered dominant-strategy implementation in a private values setting\textsuperscript{20}.

### 4.5 One-dimensional Type Spaces

The proof of Theorem 3.3 critically relied on the fact that both agents’ valuations were multi-dimensional. Under the assumption that each agent’s information is captured by one real number, positive results have been reached. Jehiel and Moldovanu (2001) and Bergemann and Välimäki (2002) both derive sufficient conditions for efficient implementation in this setting. Essentially, these authors impose an (arbitrary) order on the alternatives, and assume that each agent’s valuation function and the social surplus function are supermodular as a function of alternative and own signal\textsuperscript{21}. It is not difficult to see that this supermodularity assumption does not preclude a generic set of valuation functions. Hence, when signals are one-dimensional, ex-post implementation does not impose a knife-edge condition on valuation functions. In particular, both efficiency and incentive compatibility require higher signals to lead to (weakly) higher alternatives and efficient implementation is possible.

We next give an example with one-dimensional (actually discrete) signals, for which only trivial choice rules are implementable. This shows that some supermodularity assumption is necessary for non-trivial implementation.

#### Example 4.3

Consider a common-value, one-object allocation problem with two agents $i \in \{1, 2\}$, and discrete types $s^i \in \{-1, 1\}$. Agent $i$ values the object at $v^i(s^i, s^{\overline{i}}) = s^i s^{\overline{i}}$. Then, only trivial allocation rules are implementable.

**Proof.** Fix an implementable choice rule\textsuperscript{22} $\psi : \{-1, 1\}^2 \rightarrow \{i, j\}$ and assume w.l.o.g. that $\psi(1, 1) = j$. We must show that $\psi \equiv j$. Applying

\textsuperscript{20}Implementation with interdependent but separable valuations is closely related to dominant-strategy implementation with private values. Meyer-ter-Vehn and Moldovanu (2003) show that a choice rule $\psi$ is implementable for separable valuations, if and only if it is implementable for private valuations, i.e. omitting the terms $h_i^k$.

\textsuperscript{21}Note, that even the notion of supermodularity does not carry over well to multi-dimensional types, as it requires a natural order on the type space.

\textsuperscript{22}We slightly abuse notation here by identifying the social alternatives with the two agents.
Lemma 2.2 to agent $i$ shows\(^{23}\) that $\psi(-1, 1) = j$. Similarly, $\psi(-1, 1) = j$ and Lemma 2.2 (applied to agent $j$) yields $\psi(-1, -1) = j$. A final application of Lemma 2.2 yields $\psi(1, -1) = j$. Thus $\psi$ always allocates the good to agent $j$, and hence it is trivial. This argument is illustrated in Figure 1.

This example generalizes to continuous type spaces and to stochastic choice rules. One can also explicitly check that, as in example 3.1, the second-price auction does not have a separating ex-post equilibrium.

More generally, following the above intuition, we develop a general notion of cycling for mechanism design settings with quasi-linear utility (see Appendix B). If a setting fulfills the cycling condition, non-trivial implementation is impossible.

5 Back to Bayes-Nash Implementation?

The generic non-existence of interesting ex-post equilibria should redirect our attention to weaker equilibrium concepts. This is particularly important for applications (such as auctions) where the outcome of various allocation procedures must be somehow predicted and assessed. It is important to note that Bayes-Nash equilibria leading to non-trivial allocation rules often exist.

\(^{23}\)It cannot be that that bidder 1 gets the object when her information implies a low value, but does not get it when her information implies a high value.
even if ex-post equilibria do not. Let us revisit Example 3.1:

**Example 5.1 (Example 3.1 continued)** Assume that the signals \((p^i, c^i)\) are uniformly and independently distributed on \([0, 1]^2\). The second-price auction has a symmetric, Bayes-Nash equilibrium with a non-trivial associated allocation rule

**Proof.** We construct a symmetric BNE in continuous, strictly monotonic increasing bid functions \(b : [0, 1]^2 \rightarrow \mathbb{R}\). A necessary condition for equilibrium is that each type \((p, c)\) is indifferent between winning or losing the auction at a tie. This gives the usual condition:

\[
b = b(s^i) = \mathbb{E}_{s^{-i}}[v^i(s^i, s^{-i}) | b(s^{-i}) = b(s^i)]
\]

(8)

Given signals’ independence, the right hand side is equal to \(p^i + c^i \tau(b)\) where \(\tau(x) = \mathbb{E}_{s^{-i}}[c | b(p, c) = x]\) is the expectation of the opponent’s common values signal given that he makes a bid \(x\). This shows that the iso-bid curves \(b^{-1}(x)\) must be straight lines with slope \(-\frac{1}{c^i(\tau(x))}\) in the \((p, c)\)-space \([0, 1]^2\).

Some tedious calculations show then that the iso-bid lines are as follows:

\[
b^{-1}(x) := \begin{cases} 
(0, \sqrt{2x}), (x, 0) & \text{for } x \in [0, \frac{1}{2}] \\
(x - \frac{1}{2}, 1), (x, 0) & \text{for } x \in \left[\frac{1}{2}, 1\right] \\
(x - \frac{1}{4} - \frac{1}{4} \sqrt{8x - 7}, 1), (1, \frac{1}{2} \sqrt{8x - 7} - \frac{1}{2}) & \text{for } x \in [1, 2]
\end{cases}
\]

(9)

These iso-bid lines are drawn in figure 2:

In fact, the above displayed strategies satisfy a stronger incentive property than the one required by Bayes-Nash equilibrium: no bidder ever regrets her action (here a bid), even after learning her opponent’s action. Indeed, bidder \(i\)’s equilibrium bid \(b^i(p^i, c^i)\) beats her opponent’s equilibrium bid \(b^j\), if and only if the value of the object, in expectation over opponent’s types bidding \(b^j\), exceeds \(b^j\). In general, such a condition yields a weaker requirement than ex-post implementation since the latter requires regret-freedom with respect to the opponents’ signals. This notion of equilibrium was called *posterior implementation* by Green and Laffont (1987) who introduced it for social choice problems with two alternatives and private values. In search for weaker concepts than ex-post equilibrium, posterior implementation\(^{24}\) may be sometimes a suitable candidate. ■

\(^{24}\)Note that the information partition needed for this notion of ”regret-free equilibrium” is not defined by the design setting. Rather, it depends on the precise features of the used mechanism.
Appendix A: Proofs

Proof of Lemma 2.1. "if": Given \( t^i_k(s^{-i}) \) such that equation (2) holds, define \( t^i(s) := t^i_{\psi(s)}(s^{-i}) \). Agent \( i \)'s problem in the game induced by \((\psi, t)\) is to choose \( s^i \) in order to maximize \( v^i_{\psi(s^i, s^{-i})}(s^i, s^{-i}) - t^i_{\psi(s^i, s^{-i})}(s^{-i}) \). By equation (2), it is optimal for her to report truthfully \( s^i = s^i \), and let the choice rule \( \psi \) pick her most preferred alternative.

"only if": Let \((\psi, t)\) be an ex-post incentive compatible mechanism. We define
\[
t^i_k(s^{-i}) := \begin{cases} t^i(s^i, s^{-i}) & \text{if } \psi(s^i, s^{-i}) = k \\ \infty & \text{if } \psi(s^i, s^{-i}) \neq k \text{ for all } s^i \in S^i. \end{cases}
\] (10)
Note that \( t^i_k(s^{-i}) \) is well-defined. By \( i \)'s incentive constraint
\[
\psi(s^i, s^{-i}) = \psi(s^i, s^{-i}) = k \implies t^i(s^i, s^{-i}) = t^i(s^i, s^{-i}).
\]
By \( i \)'s incentive constraint again, she always reports in order to maximize her payoff. Thus, with \( t^i_k(s^{-i}) \) as defined in (10), condition (2) is satisfied.

Proof of Lemma 2.2. By the taxation principle, there are transfers \( t^i_k(s^i) \) such that \( \psi(s) \in \arg \max_{k'} \{ v^i_{k'}(s) - t^i_{k'}(s^{-i}) \} \) for all \( s \). If alternative \( k \) is among \( i \)'s preferred alternatives at signal \( s \), we have \( k \in \)
arg max_{k'} \{v_k^i(s) - t_{k'}^i(s^{-i})\}. If the change from \(s^i\) to \(s'^i\) makes alternative \(k\) strictly better for \(i\) than \(l\) (this means \(v_k^i(s') - v_k^i(s) > v_l^i(s') - v_l^i(s)\)), it is immediate that \(l\) can not be preferred at signal \(s'^i\): \(l \notin \arg \max_{k'} \{v_k^i(s') - t_{k'}^i(s^{-i})\}\). By the taxation principle, we conclude that \(\psi(s) \neq l\).

**Proof of Lemma 6.1.** 1) \(\mu^i(\hat{s}) - \tau^i(\hat{s}') > 0\) and \(\nabla_{s'} \mu^i(\hat{s}) \neq 0\) imply that there are \(s'^i, s''^i\) arbitrarily close to \(\hat{s}'\) such that \(\mu^i(s'^i, \hat{s}') - \tau^i(\hat{s}') < 0 < \mu^i(s''^i, \hat{s}') - \tau^i(\hat{s}')\). Applying the taxation principle to agent \(i\) yields \(\psi(s'^i, \hat{s}') = k\) and \(\psi(s''^i, \hat{s}') = l\). Hence \(\hat{s} \in I\).

For the converse, assume that \(\mu^i(\hat{s}) - \tau^i(\hat{s}') > 0\), say. By continuity, we have \(\mu^i(s) - \tau^i(s) > 0\), and thus \(\psi(s) = k\), for all \(s\) in a neighborhood of \(\hat{s}\). Thus, \(\hat{s} \notin I\).

2) By part 1, we have \(I^i(\hat{s}) = \left\{ s \in \hat{S} : s^i = \hat{s}' \text{ and } \mu^j(s) = \mu^j(\hat{s}) \right\}\). Since we assumed that \(\nabla_{s'} \mu^j \neq 0\), we can apply the implicit function theorem to conclude.

To prove Proposition 3.6, we first state a simple Lemma.

**Lemma 6.1** Let \(\phi\) and \(\xi\) be smooth functions on an open set \(X \subset \mathbb{R}^N\). Assume that there exists \(x \in X\) such that \(\phi(x) = \xi(x) = 0\), but \(\nabla \phi(x)\) and \(\nabla \xi(x)\) are not co-directional. Then there exists \(x'\) arbitrarily close to \(x\) such that \(\phi(x') < 0 < \xi(x')\).

**Proof.** As \(\nabla \phi(x)\) and \(\nabla \xi(x)\) are not co-directional, there exists a direction \(\rho \in \mathbb{R}^N\) with \(\rho \cdot \nabla \phi(x) < 0 < \rho \cdot \nabla \xi(x)\). For \(x' = x + \varepsilon \rho\), with \(\varepsilon > 0\), we get \(\phi(x') < 0 < \xi(x')\), as desired. This argument is illustrated in Figure 3.

**Proof of Proposition 3.6.** Consider an ex-post incentive compatible mechanism \((\psi, t)\) and the associated relative valuations and transfers.

1) When relative transfers are continuous, we obtain by Lemma 3.5 that

\[ \mu^i(s^i, s^j) - \tau^i(s^j) = 0 \iff s \in I. \]

Thus, agent 1 is indifferent at \(s\), if and only if this is also the case for agent 2.

Take \(\hat{s} \in I\), and assume first that \(\tau^j\) is differentiable at \(\hat{s}'\). We now apply Lemma 6.1 to \(\mu^i(\cdot, s^j) - \tau^i(s^j)\) and \(\mu^j(\cdot, s^j) - \tau^j(\cdot)\) as functions of \(s^i\) around \(\hat{s}'\), where \(s^j\) is such that \((\hat{s}'^j, s^j) \in I^j(\hat{s})\). If, contrary to the Proposition, the gradients of these functions are not co-directional, there exists \(s = (s^i, s^j)\) such that \(\mu^i(s) - \tau^i(s^j) < 0 < \mu^j(s) - \tau^j(s^j)\). This yields a contradiction.
Figure 3: If the gradients of $\phi$ and $\xi$ are not co-directional at $x$, the functions disagree at some $x'$, i.e. $\phi(x') < 0 < \xi(x')$.

to the taxation principle for ex-post equilibria. This completes the proof for the case of differentiable relative transfer functions $\tau$.

More generally, we need to deal with two sub-cases:

1. The direction of the gradient $\nabla_s \mu^i(s)$ does not depend on $s \in I^i(\hat{s})$. Instead of showing that $\tau^j$ is differentiable, we directly construct the vector $y$. Denote the orthogonal complement of $\nabla_s \mu^i(s)$ by $(\nabla \mu^i)^\perp \subset \mathbb{R}^d$ and let $\rho \in (\nabla \mu^i)^\perp$. Fix for a moment $s^j$ with $(\hat{s}^j, s^j) \in I^j(\hat{s})$. By Lemma 3.5, $\mu^j(_{s^j}, s^j) - \tau^j(\cdot)$ must equal zero on the submanifold \{ $s^i : \mu^i(s^i, s^j) = \mu^i(\hat{s}^j, s^j)$ \}. Thus, restricted to that manifold, $\tau^j$ is differentiable and we have $\partial_\rho \mu^j(\hat{s}^j, s^j) = \partial_\rho \tau^j(\hat{s}^j)$. Therefore, $\rho \cdot (\nabla_{s^j} \mu^j(\hat{s}^j, s^j) - y) = 0$ for $\rho \in (\nabla \mu^i)^\perp$. By Choosing $\lambda$ sufficiently $\nabla_{s^j} \mu^i(s)$ and $(\nabla_{s^j} \mu^j(\hat{s}^j, s^j) - y)$ must be co-directional, and condition (3) is satisfied.

2. The direction of the gradient $\nabla_{s^j} \mu^j(s)$ varies in $s \in I^i(\hat{s})$. In this case we will show that $\tau^j$ is differentiable at some $\hat{s}^i$ close to $\hat{s}^i$.

As a first step, we show that the directional derivatives $\partial_\rho \tau^j(\hat{s}^i)$ in directions $\rho \in \nabla_{s^j} \mu^j(\hat{s}^i, s^j)^\perp$ exist. Fix $s \in I^i(\hat{s})$ and $\rho \in \nabla_{s^j} \mu^j(s)^\perp$ such that there are $\xi, \eta \in I^i(\hat{s})$ close to $s$ with $\rho \cdot \nabla_{s^j} \mu^i(\xi) > 0 > \rho \cdot \nabla_{s^j} \mu^i(\eta)$. By agent $i$’s incentive constraint, we have $\psi(\hat{s}^i + \varepsilon \rho, \eta) = k$
Figure 4: An illustration of $S^i \subset \mathbb{R}^2$: the directional derivatives $\partial_\rho \tau^j(\bar{s}^i)$ exist for directions $\rho$ inside the cone. As $\nabla_{s^i} u^i(s^i, s^j)$ is continuous in $s^i$, these directional derivatives also exist in a neighborhood $U$ of $\bar{s}^i$ and are continuous.

and $\psi(\bar{s}^i + \varepsilon \rho, \bar{s}^j) = l$ for small enough $\varepsilon > 0$ (compare this argument to the one for Lemma 6.1). In turn, agent $j$’s incentive constraint implies $\partial_\rho \mu^j(\bar{s}) \geq -\frac{\tau^j(\bar{s}^i + \varepsilon \rho, \bar{s}^i)}{\varepsilon} \geq \partial_\rho \mu^j(\bar{s}^i)$. As $\bar{s}^i$ and $\bar{s}^j$ approach $s^j$, and $\varepsilon$ approaches zero, this entails $\partial_\rho \tau^j(\bar{s}^i) = \partial_\rho \mu^j(\bar{s}^i, s^j)$.

By assumption, $\nabla_{s^i} \mu^j(\bar{s}^i, s^j)$ varies (continuously) in $s^j$. Therefore, $\partial_\rho \tau^j(\bar{s}^i)$ exists for an open set of directions $\rho \in \Lambda \subset \mathbb{R}^{d^i}$. In order to conclude, we need to show that these directional derivatives are continuous in $s^i$.

Consider $\bar{s}^i = \tilde{s}^i + \varepsilon \rho$ for some $\rho \in \Lambda$ and $\varepsilon \in \mathbb{R}$ sufficiently small. By the above argument, there is a neighborhood $U$ of $\bar{s}^i$, such that the directional derivatives $\partial_\rho \tau^j(s^i)$ for $\rho \in \Lambda \subset \mathbb{R}^{d^i}$ and $s^i \in U$ exist and are continuous in $s^i$. Thus, $\tau^j$ is differentiable for $s^i \in U$ and, after replacing $\tilde{s}^i$ by $s^i$, we can conclude. For an intuition consider Figure 4.

2) Assume now that relative transfer $\tau^j$ is discontinuous at $\bar{s}^i \in \hat{S}^i$. We can assume w.l.o.g. that$^{25}$ $\tau^j(s^i) \in T^j(s^i) := \text{inf}_{s^j} \{\mu^j(s^i, s^j)\}$,

$^{25}$If $\tau^{-i}(s^i) < \text{inf}_{s^{-i}} \{v^{-i}(s^i, s^{-i})\}$, say, we have $0 < v^j(s^i, s^{-i}) + \tau^{-i}(s^i)$ for all $s^{-i}$, and agent $-i$ will "choose" outcome $k$, no matter what her signal $s^{-i}$ is. This is still the case, after we change $\tau^{-i}(s^i)$ to $\text{inf}_{s^{-i}} \{v^{-i}(s^i, s^{-i})\}$. 

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sup_{s^j} \{µ^i (s^i, s^j)\}] for all s^i. By assumption, there is a sequence of i’s signals \( (s^i_n)_{n \in \mathbb{N}} \) such that \( \lim_n s^i_n = \bar{s} \) but such that \( \tau^j (s^i_n) \) does not converge to \( \tau^j (\bar{s}) \). Modulo taking a subsequence, we can assume that \( \lim_n \tau^j (s^i_n) = \tau^j (\bar{s}) + \varepsilon \), for \( \varepsilon > 0 \), say. Consider:\(^2^6\) \( \tilde{S}^j := \{ s^j \in S^j : \mu^j (\bar{s}^j, s^j) \in \{ \tau^j (\bar{s}^j) + \varepsilon, \tau^j (\bar{s}^j) + \frac{\varepsilon}{2} \} \} \). These types \( s^j \in \tilde{S}^j \) of agent \( j \) prefer \( k \) when the relative payment is \( \tau^j (\bar{s}^j) \), but prefer \( l \) when the relative payment is \( \tau^j (\bar{s}^j) + \varepsilon \). Therefore, \( \psi (\bar{s}^j, s^j) = k \), but \( \psi (s^i_n, s^j) = l \) for large enough\(^2^7\) \( n \). As \( \lim_n s^i_n = \tilde{s} \), we can apply the taxation principle to agent \( i \) to obtain \( \mu^i (\tilde{s}^i, s^j) - \tau^i (s^j) = 0 \), for all \( s^j \in \tilde{S}^j \) (recall that \( \mu^i \) is continuous).

We now show that the gradients \( \nabla_s \mu^i (\bar{s}^j, s^j) \) are co-directional for all \( s^j \in \tilde{S}^j \). This proves the desired result since \( \tilde{S}^j \) is open, and since it contains the manifolds \( \{ s = (\bar{s}^j, s^j) : \mu^j (s) = \tau^j (\bar{s}^j) + \frac{\varepsilon}{2} \} \).

Assume that this is not the case for \( s^{i,j}, s'^{i,j} \in \tilde{S}^j (\varepsilon) \). We assume w.l.o.g. that \( \mu^i (\bar{s}^j, s^{i,j}) < \mu^i (\bar{s}^j, s'^{i,j}) \). By Lemma 6.1, there is \( \tilde{s} \), arbitrarily close to \( \bar{s} \), with \( \mu^i (\tilde{s}^j, s^{i,j}) + \tau^i (s'^{i,j}) < 0 < \mu^i (\tilde{s}^j, s^{i,j}) + \tau^i (s^{i,j}) \). Thus, \( \psi (\tilde{s}^j, s^{i,j}) = k \) and \( \psi (\tilde{s}^j, s'^{i,j}) = l \). However, for \( \tilde{s} \) close enough to \( \bar{s} \), continuity of \( \mu^i \) yields \( \mu^i (\bar{s}^j, s^{i,j}) < \mu^i (\bar{s}^j, s'^{i,j}) \). This yields a contradiction to the monotonicity of \( \psi \) and concludes the argument.

6.1 Appendix B: Cycling

In this appendix we show a way of concatenating the monotonicity constraints implied by Lemma 2.2. This leads to a technical condition on \( v \), under which only trivial choice rules are implementable.

Definition 6.2 Let \( \succeq \) be an arbitrary order on \( K \) and consider a mechanism design setting \((\mathcal{N}, K, S, v)\). We define a relation \( \succeq \) on the space of signal combinations \( S \): Set \( s' \succeq s \), if \( s \) and \( s' \) differ only in the signal of one agent \( i \), i.e. \( s^{-i} = s'^{-i} \), and if the change from \( s \) to \( s' \) makes higher alternatives more preferable to \( i \) than lower ones, i.e. \( v_k (s') - v_k (s) > v_k (s') - v_k (s) \) for \( k > l \). Let \( \succeq_S \) be the transitive closure of \( \succeq \).

Note that \( \succeq_S \) is a partial order on \( S \). This definition is geared towards the following result.

Lemma 6.3 An implementable choice rule \( \psi : S \to K \) must be weakly increasing with respect to \( \succeq_S \) and \( \succeq \).

\(^2^6\)Note that \( S^i (\varepsilon) \) is not empty. Taking \( \tau^{-i} (s^i_n) \in T^{-i} (s^i_n) \) to the limit, yields that \( \tau^{-i} (\bar{s}^i) + \varepsilon \in T^{-i} (\bar{s}^i) \). Together with \( \tau^{-i} (\bar{s}^i) \in T^{-i} (\bar{s}^i) \), this yields \( [\tau^{-i} (\bar{s}^i), \tau^{-i} (\bar{s}^i) + \varepsilon] \subset T^{-i} (\bar{s}^i) = [\inf_{s^{-i}} \{ v^{-i} (s^i, s^{-i}) \}, \sup_{s^{-i}} \{ v^{-i} (s^i, s^{-i}) \}] \).

\(^2^7\)That is, \( n \) such that: \( v^{-i} (s^i_n, s^{-i}) < v^{-i} (\bar{s}^i, s^{-i}) + \frac{\varepsilon}{2} \leq \tau^{-i} (\bar{s}^i) + \frac{\varepsilon}{4} < \tau^{-i} (s^i_n) \)
Proof. By the monotonicity of \( \psi \) we have that \( \psi(s') \succeq \psi(s) \) if \( s' \succeq s \). This carries over to the transitive closure \( \succeq_{S^o} \) of \( \succeq \).

The order \( \succeq \) on \( K \) is arbitrary and has no straightforward interpretation. It is a technical device to make the monotonicity constraint of \( \psi \) carry over to concatenations of signal changes.

**Definition 6.4** A subset \( S' \subset S \) is an improvement cycle, if there is an order \( \succeq \) on \( K \) such that \( S' \) is an equivalence class of \( \succeq_S \), i.e. \( s \succeq_S s' \succeq_S s \) for all \( s, s' \in S' \). A mechanism design setting \( (N, K, S, v) \) exhibits strong cycling if the whole signal space \( S \) is an improvement cycle.

By Lemma 6.3, an implementable choice rule \( \psi \) must be constant on improvement cycles, yielding the following proposition.

**Proposition 6.5** If a mechanism design setting \( (N, K, S, v) \) exhibits strong cycling, only trivial choice rules are implementable.

For a setting with strong cycling, it follows that any ex-post equilibrium of any mechanism must be a complete pooling equilibrium.

The setting of Example 4.3 exhibits strong cycling. Ordering the alternatives "1 gets the object" \( \succeq \) "2 gets the object", it is easy to verify, that \( (1,1) \succeq (-1,1) \succeq (-1,-1) \succeq (1,-1) \succeq (1,1) \). Thus, \( S = \{-1,1\}^2 \) is an improvement cycle. Applying Proposition 6.5, we get that only trivial allocation rules are implementable.

**References**


Appendix C: Generic Maps and Differential Equations

by William R. Zame

The main text shows that ex-post implementability implies that pairs of smooth valuation functions satisfy a first-order differential equation (with constant coefficients) on the intersection of a level set with a coordinate subspace. This Appendix shows that generic pairs of smooth functions do not have this property, and hence that ex-post implementability is generically impossible.

The proof is based on the Multi-Jet Transversality Theorem (a far-reaching generalization of the ordinary Transversality Theorem), and makes use of some facts about semi-algebraic sets. Because these are perhaps not part of the usual tool-kit, the necessary background is collected below. Because it involves only slightly more effort, and more clearly demonstrates the power of the method, more is established than is needed: generic pairs of smooth functions do not satisfy any polynomial differential equation (with constant coefficients) on the intersection of a level set with a coordinate subspace. The main result is established for pairs of $C^\infty$ functions defined on an open set; a Corollary derives a parallel result for pairs of $C^1$ functions defined on a compact set. Similar results for tuples of functions of various differentiability are easily obtained by small variations on the arguments given.

Some notation is necessary. Fix an integer $d \geq 2$ and an open subset $\Omega \subset \mathbb{R}^d$. Write $C^\infty(\Omega, \mathbb{R})$ for the space of functions $f : \Omega \to \mathbb{R}$ which admit continuous derivatives of all orders, and $C^\infty(\Omega, \mathbb{R}^2) = C^\infty(\Omega, \mathbb{R}) \times C^\infty(\Omega, \mathbb{R})$ for the space of maps $(f, g) : \Omega \to \mathbb{R}$ for which $f, g \in C^\infty(\Omega, \mathbb{R})$. A multi-index is a $d$-tuple $\alpha = (\alpha_1, \ldots, \alpha_d)$ of non-negative integers; the order of the multi-index $\alpha$ is $|\alpha| = \alpha_1 + \ldots + \alpha_d$. If $f \in C^\infty(\Omega, \mathbb{R})$ and $\alpha$ is a multi-index, write

$$\partial^\alpha f = \frac{\partial^{\alpha_1 + \ldots + \alpha_d} f}{\partial \omega_1^{\alpha_1} \cdots \partial \omega_d^{\alpha_d}}$$

Write 0 for the multi-index $(0, \ldots, 0)$; by convention, $\partial^0 f = f$.

Equip $C^\infty(\Omega, \mathbb{R})$ and $C^\infty(\Omega, \mathbb{R}^2)$ with the Whitney topology. A sub-base for this topology on $C^\infty(\Omega, \mathbb{R})$ is given by sets of the form

$$V(f, k, \epsilon) = \{g \in C^\infty(\Omega, \mathbb{R}) : |\partial^\alpha f(\omega) - \partial^\alpha g(\omega)| < \delta(\omega)\}$$

for every $\omega \in \Omega$, every $|\alpha| \leq k$.

where $f \in C^\infty(\Omega, \mathbb{R})$, $k$ is an integer, and $\delta : \Omega \to \mathbb{R}^+_+$ is a continuous function. In the Whitney topology, $C^\infty(\Omega, \mathbb{R})$ and $C^\infty(\Omega, \mathbb{R}^2)$ are Baire.
spaces; that is, residual sets (intersections of countable families of dense open sets) are dense. (Golubitsky and Guillemin (1986) is a convenient reference.)

For $k$ a non-negative integer, write $I(k)$ for the set of multi-indices of order at most $k$. A polynomial in the variables $x_\alpha, y_\beta; \alpha, \beta \in I(k)$ is a sum of the form

$$P = \sum_{A,B} c(A,B) \prod_{\alpha \in I(k)} x_\alpha^{a(\alpha)} \prod_{\beta \in I(k)} y_\beta^{b(\beta)}$$

where the summation extends over all arrays $A = (a(\alpha)), B = (b(\beta))$ of non-negative integers, each coefficient $c(A,B)$ is a real number, and only finitely many of the coefficients $c(A,B)$ are non-zero. The order of $P$ is the highest order of any multi-index that appears in a term with a non-zero coefficient; the degree of $P$ is the highest exponent that appears in a term with a non-zero coefficient. For such a polynomial $P$, the polynomial differential operator $P(D)$ is defined by substituting $\partial^\alpha f$ for $x_\alpha$ and $\partial^\beta g$ for $y_\beta$; that is

$$P(D)(f,g) = \sum_{A,B} c(A,B) \prod_{\alpha \in I(k)} (\partial^\alpha f)^{a(\alpha)} \prod_{\beta \in I(k)} (\partial^\beta g)^{b(\beta)}$$

(Recall that $\partial^0 f = f$.) By the order and degree of the operator $P(D)$ are meant the order and degree of the polynomial $P$.

Fix an integer $d_1, 0 \leq d_1 \leq d - 2$. For $z \in \mathbb{R}^{d_1}, g \in C^\infty(\Omega, \mathbb{R}), c \in \mathbb{R}$, write

$$I(z, g, c) = \{ \omega \in \Omega : (\omega_1, \ldots, \omega_{d_1}) = z \} \cap \{ \omega \in \Omega : g(\omega) = c \}$$

(If $d_1 = 0$ then $\mathbb{R}^{d_1} = \{0\}$ and the condition $(\omega_1, \ldots, \omega_{d_1}) = z$ is vacuously satisfied.) For lack of a better term, say $P$ (or $P(D)$) is degenerate at $c \in \mathbb{R}$ if $P(x,y) = 0$ whenever $y_0 = c$; otherwise $P$ is non-degenerate at $c$. $P(D)$ is degenerate at $c$ if and only if $P(D)(f,g)(\omega) = 0$ for every $(f,g), \omega$ for which $g(\omega) = c$.

The main result is the following theorem:

**Theorem A:** There is a residual subset $H \subset C^\infty(\Omega, \mathbb{R}^2)$ such that: if $(f,g) \in H, z \in \mathbb{R}^{d_1}, c \in \mathbb{R}$ and $P(D)$ is a polynomial differential operator that is non-degenerate at $c$, then $P(D)(f,g)$ does not vanish on any subset of $I(z, g, c)$ that has dimension at least $d - d_1 - 1$.\(^{28}\)

**Corollary 1:** Let $G_1 \subset C^\infty(\Omega, \mathbb{R}^2)$ be the open set of maps $(f,g)$ for which neither $(\partial f/\partial \omega_1, \ldots, \partial f/\partial \omega_{d_1})$ nor $(\partial g/\partial \omega_{d_1+1}, \ldots, \partial g/\partial \omega_d)$ vanish

\(^{28}\)The restriction to non-degenerate polynomial differential operators is of course necessary: if $P(D)$ is degenerate at $c$ then $P(D)(f,g)$ vanishes identically on $\{ \omega \in \Omega : g(\omega) = c \}$. 25
at any point of \( \Omega \). There is a residual subset \( H_1 \subset G_1 \) such that: if \((f,g) \in H_1, z \in IR^d, c \in IR, P(D) \) is a polynomial differential operator that is non-degenerate at \( c \), and \( I(z,g,c) \neq \emptyset \), then \( P(D)(f,g) \) does not vanish identically on \( I(z,g,c) \).

The results above are formulated for valuation functions that are infinitely differentiable and defined on an open domain, but similar results for valuation functions that are less differentiable and/or defined on a compact domain are also easily obtained as corollaries. The precise result necessary for the generic impossibility of ex-post implementation follows:

Consider a compact set \( K \subset IR^d \). Let \( C^1(K, IR^2) \) be the space of maps \( K \to IR^2 \) that admit continuously differentiable extensions to an open neighborhood of \( K \). Equipped with the topology of uniform convergence of maps and first derivatives, \( C^1(K, IR^2) \) is a Banach space, hence a Baire space. (See Malgrange (1966).)

**Corollary 2:** Let \( K \subset IR^d \) be a compact set with non-empty interior \( \text{int} K \) and let \( G_2 \subset C^1(K, IR^2) \) be the open subset of mappings \((f,g) \) such that \((\partial f/\partial \omega_1, \ldots, \partial f/\partial \omega_d) \) and \((\partial g/\partial \omega_{d+1}, \ldots, \partial g/\partial \omega_d) \) do not vanish at any point of \( K \). There is a residual subset \( H_2 \subset G_2 \) such that: if \((f,g) \in H_2, z \in IR^d, c \in IR, P(D) \) is a polynomial differential operator of order at most 1 that is non-degenerate at \( c \), and \( I(z,g,c) \cap \text{int} K \neq \emptyset \), then \( P(D)(f,g) \) does not vanish identically on \( I(z,g,c) \).

Before beginning the proofs, recall some ideas about jet bundles and multi-jet bundles. (Again, Golubitsky and Guillemin (1986) is a convenient reference.) For \( k \) a positive integer, write \( IR^{I(k)} \) for the vector space of functions \( I(k) \to IR \). (Note that \( IR^{I(k)} \) is isomorphic to the Euclidean space \( IR^{|I(k)|} \).) The \( k \)-jet bundles are the total spaces and projections:

\[
\pi : J_k(\Omega, IR) = \Omega \times IR^{I(k)} \to \Omega \\
\pi : J_k(\Omega, IR^2) = \Omega \times IR^{I(k)} \times IR^{I(k)} \to \Omega
\]

For \( f \in C^\infty(\Omega, IR), \omega \in \Omega \), write

\[
T_k f(\omega) = (f_\alpha(\omega))_{\alpha \in I(k)}
\]

That is, \( T_k f(\omega) \) is a list of the values at \( \omega \) of \( f \) and all partials through order \( k \). Identify \( T_k f(\omega) \) as an element of \( IR^{I(k)} \) in the obvious way. Define the \( k \)-jet of \( f \) at \( \omega \) to be

\[
j_k f(\omega) = (\omega, T_k f(\omega)) \in J_k(\Omega, IR)
\]
Similarly, if \((f, g) \in C^\infty(\Omega, \mathbb{R}^2)\), the \(k\)-jet of \((f, g)\) at \(\omega\) is
\[
j_k(f, g)(\omega) = (\omega, T_k f(\omega), T_k g(\omega)) \in J_k(\Omega, \mathbb{R}^2)
\]
Note that
\[
j_k f : \Omega \to J_k(\Omega, \mathbb{R})
j_k (f, g) : \Omega \to J_k(\Omega, \mathbb{R}^2)
\]
are sections of the respective jet bundles. It is convenient to adopt the obvious coordinates \(\omega; x \in J_k(\omega, \mathbb{R})\) and \(\omega; x, y \in J_k(\Omega, \mathbb{R}^2)\) so that \(x_\alpha(j_k f(\omega)) = \partial^\alpha f(\omega)\) etc.

For \(r \geq 1\), write
\[
\pi^r : J_k^r(\Omega, \mathbb{R}^2)^r \to \Omega^r
\]
for the \(r\)-fold product of the projection \(\pi : J_k(\Omega, \mathbb{R}^2) \to \Omega\). Let \(\Omega^{(r)} \subset \Omega^r\) be the open subset of distinct \(r\)-tuples. The \(r\)-fold \(k\)-multi-jet bundle is
\[
\pi^r : J_k^r(\Omega, \mathbb{R}^2)^r \to \Omega^{(r)}
\]
For \((f, g) \in C^\infty(\Omega, \mathbb{R}^2)\) and \((\omega^1, \ldots, \omega^r) \in \Omega^{(r)}\), the \(r\)-fold \(k\)-multi-jet of \((f, g)\) at \((\omega^1, \ldots, \omega^r)\) is
\[
j_k^r(f, g)(\omega^1, \ldots, \omega^r) = (j_k(f, g)(\omega^1), \ldots, j_k(f, g)(\omega^r))
\]
\[
= (\omega^1, \ldots, \omega^r; T_k f(\omega^1), \ldots, T_k f(\omega^r); T_k g(\omega^1), \ldots, T_k g(\omega^r))
\]
Note that \(j_k^r(f, g)\) is a section of \(J_k^r(\Omega, \mathbb{R}^2)\), and that the image \(j_k^r(f, g)(\Omega^{(r)})\) is a submanifold of \(J_k^r(\Omega, \mathbb{R}^2)\).

If \(Z\) is a manifold, \(W, W' \subset Z\) are submanifolds, and \(z \in W \cap W'\) is a point of the intersection, say \(W, W'\) are transversal at \(z\) if the tangent spaces to \(W\) and \(W'\) at \(z\) span the tangent space to \(Z\) at \(z\). Say \(W, W'\) are transversal if they are transversal at every point at which they intersect. If \((f, g) \in C^\infty(\Omega, \mathbb{R}^2)\) and \(W \subset J_k^r(\Omega, \mathbb{R}^2)\) is a submanifold, say \((f, g)\) is transversal to \(W\) if the map \(j_k^r(f, g)(\Omega^{(r)})\) is transversal to \(W\).

**Multi-Jet Transversality Theorem:** If \(W \subset J_k^r(\Omega, \mathbb{R}^2)\) is a submanifold then \(\{(f, g) \in C^\infty(\Omega, \mathbb{R}^2) : (f, g)\) is transversal to \(W\}\) is a residual subset of \(C^\infty(\Omega, \mathbb{R}^2)\).

In the proof of Theorem A, the Multi-Jet Transversality Theorem is applied to a set \(W\) that need not be a manifold. To accomplish this end, the theory of semi-algebraic sets is used to show that the set in question is a finite
union of manifolds. The basic facts needed are given below. (See Bochnak, Coste and Roy (1986) for a good general reference, or Schanuel, Simon and Zame (1991) and Blume and Zame (1994) for expositions in a game-theoretic setting.)

A subset $A \subset \mathbb{R}^m$ is semi-algebraic if it is the finite union of sets defined by a finite number of polynomial equalities and inequalities; that is, a finite union of sets of the form:

\[
\{ z \in \mathbb{R}^m : f_1(z) = 0, \ldots , f_I(z) = 0, g_1(z) \leq 0, \ldots , g_J(z) \leq 0, h_1(z) < 0, \ldots , h_M(z) < 0 \}
\]

where each $f_i$, $g_j$, $h_m$ is a real polynomial. Note that the finite intersection of semi-algebraic sets is semi-algebraic. Semi-algebraic sets have a very special structure; the properties relevant here are:

a) Every semi-algebraic set is the finite union of smooth manifolds.

b) If $A \subset \mathbb{R}^m$ is semi-algebraic and $\Pi : \mathbb{R}^m \to \mathbb{R}^m'$ is the projection then $\Pi(A)$ is semi-algebraic, and the dimension of $\Pi(A)$ does not exceed the dimension of $A$.\(^{29}\)

**Proof of Theorem A:** The idea is to construct, for each pair of positive integers $k, \ell$, a residual subset $H(k; \ell) \subset C^\infty(\Omega, \mathbb{R}^2)$ with the following property:

**Property:** If $(f, g) \in H(k; \ell)$, $z \in \mathbb{R}^{d_1}$, $c \in \mathbb{R}$ and $P(D)$ is a polynomial differential operator of order at most $k$ and degree at most $\ell$ that is non-degenerate at $c$, then $P(D)(f, g)$ does not vanish on any subset of $I(z, g, c)$ that has dimension at least $d - d_1 - 1$.

Once this is accomplished, construction of the desired set $H$ is almost immediate. To this end, fix positive integers $k, \ell$. Write $P(k; \ell)$ for the finite dimensional vector space of polynomials of order at most $k$ and degree at most $\ell$, and $\dim P(k; \ell)$ for its dimension. Choose an index $r$ with

\[
r > d_1 + \dim P(k; \ell) + 2|\mathcal{I}(k)|
\]

Work in the $r$-fold multi-jet bundle $J^k_r(\Omega, \mathbb{R}^2)$. Let $W \subset J^k_r(\Omega, \mathbb{R}^2)$ be the set of tuples $((\omega^i); (x^i), (y^i))$ for which there exists $c \in \mathbb{R}$, $z \in \mathbb{R}^{d_1}$, and a polynomial $P \in P(k; \ell)$ such that $P$ is non-degenerate at $c$, and for each $i$: $(\omega^i_1, \ldots , \omega^i_{d_1}) = z$, $y^i_0 = c$, and $P(x^i, y^i) = 0$.

\(^{29}\)Because semi-algebraic sets are finite unions of smooth manifolds, dimension is unambiguously defined.
W need not be a manifold, but it has a special structure, detailed in the following:

**Claim:** \( W \) is a finite union of manifolds \( W = W_1 \cup \ldots \cup W_N \), and the dimension of \( W \) (that is, the largest dimension of any of the manifolds \( W_1, \ldots, W_N \)) and co-dimension of \( W \) (that is the smallest co-dimension of any of the manifolds \( W_1, \ldots, W_N \)) satisfy the inequalities:

\[
\dim W \leq \dim J_k^r(\Omega, \mathbb{R}^2) + d_1 + \dim \mathcal{P}(k, \ell) + 2|\mathcal{I}(k)| - 2r - rd_1
\]
\[
\text{codim} W \geq 2r + rd_1 - d_1 - \dim \mathcal{P}(k, \ell) - 2|\mathcal{I}(k)|
\]

To establish the Claim, define

\[
W^* = \{ (\omega, x, y, c, z, P, \xi, \eta) \in J_k^r(\mathbb{R}^d, \mathbb{R}^2) \times \mathbb{R} \times \mathbb{R}^{d_1} \times \mathcal{P}(k, \ell) \times \mathcal{P}(z) \times \mathbb{R}^{\mathcal{I}(k)} : \\
\eta_0 = c, P(\xi, \eta) \neq 0, y_j^i = c \text{ for each } j, \\
(\omega_i^1, \ldots, \omega_i^d_1) = z \text{ for each } i, \\
P(x_i^1, y_i) = 0 \text{ for each } i \}
\]

Let

\[
\Pi : J_k^r(\mathbb{R}^d, \mathbb{R}^2) \times \mathbb{R} \times \mathbb{R}^{d_1} \times \mathcal{P}(k, \ell) \rightarrow J_k^r(\mathbb{R}^d, \mathbb{R}^2)
\]

be the projection. By definition, \( W^* \) is a semi-algebraic set. Counting independent equations shows that its co-dimension satisfies

\[
\text{codim } W^* \geq 1 + 2r + rd_1
\]

whence its dimension satisfies

\[
\dim W^* \leq \dim J_k^r(\Omega, \mathbb{R}^2) + d_1 + \dim \mathcal{P}(k, \ell) + 2|\mathcal{I}(k)| - 2r - rd_1
\]

In view of the properties of semi-algebraic sets given above, it follows that \( \Pi(W^*) \) is a semi-algebraic set, hence the finite union of manifolds and

\[
\dim \Pi(W^*) \leq J_k^r(\Omega, \mathbb{R}^2) + d_1 + \dim \mathcal{P}(k, \ell) + 2|\mathcal{I}(k)| - 2r - rd_1
\]

whence

\[
\text{codim } \Pi(W^*) \geq 2r + rd_1 - d_1 - \dim \mathcal{P}(k, \ell) - 2|\mathcal{I}(k)|
\]

Note that \( W = \Pi(W^*) \cap \Omega \), so that the Claim follows immediately.
For each $i$, set $H_n(k, \ell) = \{(f, g) \in C^\infty(\Omega, \mathbb{R}^2) : (f, g) \text{ is transversal to } W_n\}$, and define

$$H(k, \ell) = \bigcap_n H_n(k, \ell)$$

The Multi-jet Transversality Theorem guarantees that each $H_n(k, \ell)$ is a residual set, so $H(k, \ell)$ is also a residual set.

To see that $H(k, \ell)$ satisfies the Property, suppose, by way of obtaining a contradiction, that $(f, g) \in H(k, \ell)$, $z \in \mathbb{R}^{d_1}$, $c \in \mathbb{R}$, $P(D)$ is a polynomial differential operator of order at most $k$ and degree at most $\ell$ that is non-degenerate at $c$, and $P(D)(f, g)$ vanishes on some subset $Z$ of $I(z, g, c)$ that has dimension at least $d - d_1 - 1$. Note first that because $(f, g) \in H(k, \ell)$, the definition of transversality guarantees that

$$E(f, g) = \{ (\omega^1, \ldots, \omega^r) \in \Omega^{(r)} : j^r_k(f, g)(\omega^1, \ldots, \omega^r) \in W \}$$

has codimension in $\Omega^{(r)}$ at least the codimension of $W$ in $J^r_k(\Omega, \mathbb{R}^2)$

$$\dim E(f, g) \geq 2r + rd_1 - d_1 - \dim \mathcal{P}(k, \ell) - 2|I(k)| \quad (11)$$

If $\omega^1, \ldots, \omega^r \in Z$ then $(\omega^1, \ldots, \omega^r) \in E(f, g)$. Hence $Z^{(r)} \subset E(f, g)$ and

$$\dim E(f, g) \geq \dim Z^{(r)} = r \dim Z = r(d - d_1 - 1) \quad (12)$$

On the other hand,

$$\dim E(f, g) + \dim E(f, g) = \dim \Omega^{(r)} = rd \quad (13)$$

Combining the equalities and inequalities (11)-(13) yields:

$$\left[ 2r + rd_1 - d_1 - \dim \mathcal{P}(k, \ell) - 2|I(k)| \right] + \left[ r(d - d_1 - 1) \right] \leq rd$$

Simplifying yields

$$r \leq d_1 + \dim \mathcal{P}(k, \ell) + 2|I(k)|$$

which contradicts the choice of $r$. It follows that $H(k, \ell)$ satisfies the Property.

Now set

$$H = \bigcap_{k, \ell=1}^{\infty} H(k, \ell)$$

30
Because it is a countable intersection of residual sets, $H$ is itself a residual set. It is evident that if $(f, g) \in H$, $z \in \mathbb{R}^{d_1}$, $c \in \mathbb{R}$ and $P(D)$ is a polynomial differential operator that is non-degenerate at $c$, then $P(D)(f, g)$ does not vanish on any subset of $I(z, g, c)$ that has dimension at least $d - d_1 - 1$, so the proof is complete.

**Proof of Corollary 1:** That $G_1$ is open follows immediately from the definition of the Whitney topology. Let $H \subset C^\infty(\Omega, \mathbb{R}^2)$ be the residual subset constructed in Theorem A, and set $H_1 = H \cap G$. If $(f, g) \in G_1$, $z \in \mathbb{R}^{d_1}$, $c \in \mathbb{R}$ and $I(z, g, c)$ is not empty then it is a manifold of dimension $d - d_1 - 1$, so the conclusion follows immediately from the Theorem.

**Proof of Corollary 2:** That $G_2$ is open is an immediate consequence of the definition of the topology in $C^1(K, \mathbb{R}^2)$. $H_2$ is constructed as the countable intersection of open subsets of $G_2$; Theorem A and use the Theorem to show that these open subsets are dense in $G$.

To this end, choose an increasing sequence $L_1, L_2, \ldots$ of compact sets whose union is $\text{int}K$. For each index $m$, let $F_2(m)$ be the set of $(f, g) \in G_2$ for which there exist $z \in \mathbb{R}^{d_1}$, $c \in \mathbb{R}$, a polynomial $P$ of order 1, and $(\xi, \eta) \in \mathbb{R}^d \times \mathbb{R}^d$ such that

- the degree of $P$ is at most $m$
- $z, c, \xi, \eta$ and all the coefficients of $P$ are bounded (in absolute value) by $m$
- $\eta_0 = c$
- $|P(\xi, \eta)| \geq 1/m$
- $P(D)(f, g)$ vanishes on some subset $V \subset I(z, g, c) \cap L_m$ having $d - d_1 - 1$-dimensional Hausdorff measure at least $1/m$

It is straightforward to see that $F_2(m)$ is a closed subset of $G_2$, whence $H_2(m) = G \setminus F_2(m)$ is open.

Fix any open set $\Omega \subset \mathbb{R}^d$ that contains $K$, and let

$$\rho : C^\infty(\Omega, \mathbb{R}^2) \to C^1(K, \mathbb{R}^2)$$

be the restriction. Note that $\rho[C^\infty(\Omega, \mathbb{R}^2)]$ is dense in $C^1(K, \mathbb{R}^2)$ (see Malgrange (1966)). Let $H \subset C^\infty(\Omega, \mathbb{R}^2)$ be the residual set constructed.

\[30] To define the $d - d_1 - 1$-dimensional Hausdorff measure of $V$, consider countable covers \{${B(v_i, b_i)}$\} of $V$ by open balls. For each such cover, consider the sum $\sum b_i^{d-d_1-1}$. The $d - d_1 - 1$-dimensional Hausdorff measure of $V$ is the infimum of all these sums.
in the Theorem. From the definition of \( G_2 \) it follows that if \((f, g) \in H\), \( \rho(f, g) \in G_2 \) and \( I(z, g, c) \cap \text{int}K \neq \emptyset \), then \( I(z, g, c) \cap \text{int}K \) is a \( d - d_1 - 1 \)-dimensional manifold, and therefore that \( \rho(f, g) \in H_2(m) \) for every \( m \). Hence \( \rho(H) \cap G_2 \subset H_2(m) \) for each \( m \). \( H \) is dense in \( C^\infty(\Omega, \mathbb{R}^2) \), so \( \rho(H) \) is dense in \( C^1(K, \mathbb{R}^2) \) and \( \rho(H) \cap G_2 \) is dense in \( G_2 \). Thus, each \( H_2(m) \) is dense in \( C^1(K, \mathbb{R}^2) \). The desired set is \( H_2 = \bigcap_m H_2(m) \).

References


