Ex-post Implementation with Interdependent Valuations

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Abstract

We consider a social choice setting with multidimensional signals and interdependent valuations. Such frameworks have been recently and increasingly used in order to study multi-object auctions. We obtain concise characterizations of ex-post implementable (not necessarily efficient) social choice functions in terms of affine functions that associate a weight to each agent and to each alternative. These choice rules can be seen as weighted extensions of the CGV mechanisms. Our characterization greatly reduces the complexity of the search for a constrained efficient (i.e., second best) mechanism in the generic cases where efficient outcomes cannot be implemented.

1 Introduction

We study a social choice problem where society has to choose among several possible alternatives. Each agent obtains a private signal about each possible alternative, and an agent’s valuation for a given alternative depends both on her own, and on the other agents’ information. An agent’s utility is given by the sum of her valuation for the chosen alternative and a monetary transfer. Thus, we consider a model with quasi-linear utility functions, multidimensional signals and interdependent valuations.

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A well-known special sub-case is the *private-values* model where an agent’s valuation for an alternative depends only on her own information. In this context, a prominent role is played by the Clarke-Groves-Vickrey (CGV) mechanisms (see Vickrey, 1961, Clarke, 1971 and Groves, 1973). These are direct revelation mechanisms where a value-maximizing (or efficient) alternative is chosen for any realization of signals, and where truthfully revealing the private signal is a dominant strategy for each agent. The key insight behind these mechanisms is that individual transfers are perfectly aligned to the marginal impact of the individual reports on society’s welfare. This alignment works because the valuations of all agents other than \(i\), say, do not depend on \(i\)’s information\(^1\). There are numerous applications of the CGV idea, which is central to mechanism design.

A growing literature (in particular in the subfield of auction theory) has departed from the restrictive informational assumption of the private values paradigm, and allowed valuations to also depend on others’ information. Non-trivial dominant strategy mechanisms usually do not exist if valuations are interdependent. In particular, the marginal impact of agent \(i\) on society’s welfare depends also on \(i\)’s signal, and CGV mechanisms which use these impacts as transfers fail to create the right incentives for truthful revelation.

Milgrom (1981) and Milgrom and Weber (1982) considered one-object auction models where each bidder’s valuation depends on all agents’ one-dimensional signals. These authors focused on Bayes-Nash equilibria of standard auction formats in the case where the agents’ valuations all have the same functional form (i.e., the model is symmetric), and where, by assumption, the agent with the highest signal has the highest value. Hence, in those models an efficient allocation is achieved by all auction formats where the agent with the highest signal gets the object.

Cremer and McLean (1985) focus their attention on the possibility of rent extraction when the agents’ signals are correlated (maximal rent extraction implies, in particular, that a value-maximizing alternative must be chosen). They allow for interdependent and asymmetric valuations while considering a finite set of linearly ordered signals for each agent. In this set-up they construct an efficient mechanism where truth telling is an *ex-post equilibrium*. The idea is to adjust the CGV transfer to agent \(i\) in a way that neutralizes the impact of \(i\)’s signal on society’s welfare (and hence on \(i\)’s transfer). The ex-post equilibrium notion is stronger than Bayes-Nash equilibrium and

\(^1\)It is also worth mentioning here that weakening the implementation requirement to Bayes-Nash equilibrium does not add much freedom: for example, if signals are independent, the expected transfers in any efficient mechanism where truth-telling is a Bayes-Nash equilibrium correspond to Clarke-Groves-Vickrey transfers. This is a consequence of the so called Revenue-Equivalence Theorem.

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weaker than dominant strategy equilibrium. For truth-telling in a direct revelation mechanism to be an ex-post equilibrium, correctly revealing the private information must be individually optimal ex-post, i.e. even after an agent has discovered the information available to others (and no matter what this information turns out to be). Roughly speaking, this is a condition of no-regret.

The Cremer-McLean construction hinges on a technical condition on valuations called *single-crossing* which restricts the set of permissible valuation functions in order to enable an alignment of private and social incentives (where the latter are represented by the sum of the individual valuations). A general formulation in a social choice framework is as follows: if a change in agent $i$’s signal improves $i$’s valuation of alternative $k$ more than $i$’s valuation of alternative $l$, then the same must be true for the relative improvement of social welfare. In the private values case this condition is trivially satisfied since the improvement in the social valuation (which depends on $i$’s signal only via $i$’s valuation) coincides with the individual improvement. This fact allows the efficient implementation via CGV mechanisms. As long as signals are one-dimensional, analogue conditions can be generically satisfied also in frameworks with interdependent valuations, and therefore ex-post incentive compatible and ex-post efficient mechanisms have been be constructed for a variety of models (see for example Ausubel (1997), Bergemann and Välimäki (2000), Dasgupta and Maskin (2000), Jehiel and Moldovanu (2001), Maskin (1991), and Perry and Reny (1999)). For example, in an one-object auction model with one-dimensional signals and interdependent valuations, single-crossing boils down to the requirement that $i$’s signal has a stronger impact on $i$’s valuation than on $j$’s valuation, $j \neq i$.

The situation changes when signals are multidimensional. Jehiel and Moldovanu (2001) (JM) consider a social choice model with multidimensional, independently drawn signals (each agent obtains a private signal about each possible alternative), and with interdependent valuations that are linear functions of the signals. In this simple framework they show that efficiency is not compatible with Bayes-Nash incentive compatibility unless a strong, non-generic condition$^2$ holds on the valuation functions$^3$. The JM condition requires that the marginal impact of an agent’s various signals on the society’s welfare is independent of the chosen alternative. In other words, efficient implementation is possible only for individual valuation functions where we can define the “weight” of each agent, independently of the chosen

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$^2$This condition is trivially satisfied in the private values case and in the pure common values case.

$^3$In auction frameworks this difficulty arises either in multi-object auctions or in one-object auctions with allocative externalities.
alternative.

The JM result immediately implies that, no matter how signals are distributed, efficiency is generally not compatible with ex-post incentive compatibility\(^4\). This impossibility result focuses the attention on other desirable and implementable social choice rules (e.g., such that may arise if we search for mechanisms that maximize welfare subject to the incentive compatibility constraints). Given that in the Bayes-Nash framework the incentive compatibility constraint takes the form of a complex partial differential equation, there is little hope for analytical advances within that framework. This in turn redirects our attention to the stronger and more robust (since independent of signals’ distributions) notion of implementation in ex-post equilibrium: a relatively simple characterization for all mechanisms in this smaller class would greatly simplify the search for second-best mechanisms in a variety of settings. Obtaining a powerful characterization is the main goal of the present paper, and we need to assume here that valuations are the sum of two components, where the first component describes the dependence on one’s own signal, and the second component describes the dependence on others’ signals. Although this semi-separability is restrictive, the advantage is that it allows a very concise and powerful characterization of ex-post implementability. As a byproduct of their study of relations between implementation concepts and potentials a la Monderer-Shapley, Jehiel et.al (2002) show that, without semi-separability, only trivial (i.e. constant) choice rules are generically ex-post implementable\(^5\).

The main role in the analysis is played by a basic property called ”positive association of differences” (PAD), which must be fulfilled by any ex-post implementable social choice rule. Roughly speaking, PAD says that if a social choice rule chooses alternative \(k\) when the signal is \(s\), then \(k\) continues to be chosen for all other signal realizations which, for all agents, make alternative \(k\) relatively more preferable than all other alternatives. By imposing conditions on individual valuations, the single crossing properties mentioned above indirectly ensure that the efficient SCR satisfies PAD. Thus, these properties were specifically tailored for the efficiency criterion. In contrast, our results show that an arbitrary SCR is implementable if and only it satisfies PAD.

The main result of this paper is Theorem 1 which characterizes all de-

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\(^4\) Otherwise, since ex-post equilibria do not depend on the distribution of signals, we would obtain efficiency also for the case of independently drawn signals contradicting the JM result.

\(^5\) Theorem 1 in Chung and Ely (2001) offers a characterization of ex-post implementability (akin to the ”taxation principle”) in the general framework. But, as the paper by Jehiel et.al. shows, the set of non-trivial implementable SCRs is generically empty without the semi-separability assumption.
terministic, ex-post implementable social choice rules: every such choice rule must maximize an affine function of the impacts the agents’ signals have on own valuations. Such a function is determined by agent-specific and alternative-independent weights with which an agent’s information enters the choice rule, and by additional alternative-specific weights that are independent of agents’ information. These choice rules can be seen as weighted extensions of the CGV mechanisms.

The proof of the above result relies on a subtle Theorem obtained by Roberts (1979). Roberts worked within a private values framework and characterized, in terms of affine functions, all social choice rules that are dominant strategy implementable (without the efficiency requirement). We are able to use Roberts’ result since its proof relies solely on the PAD property, which is satisfied for dominant strategy implementable choice rules in his case, and for ex-post implementable choice rules in ours.

Our result (as Roberts’) hinges on two main conditions: first, there must be at least three social alternatives; second, the space of valuation functions must be sufficiently rich in the sense that, for any fixed signals of others, some agent can, by varying her signal, force the choice of any alternative.

This last requirement is automatically satisfied if each agent’s signal space allows for all cardinal utility levels for all alternatives. We show via examples how the characterization fails if at least one of the conditions is not satisfied, and we also obtain a weaker characterization for the case with only two alternatives (this resembles a result by Laffont and Maskin, 1982 for the private values case).

We focus next on efficient implementation. We show that the JM necessary condition for efficient Bayes-Nash implementation is in fact necessary and sufficient for ex-post implementation. Thereby we show that the a priori weaker requirement of efficient Bayes-Nash implementation with independent signals is in fact equivalent to efficient ex-post implementation.

The paper is organized as follows: In Section 2 we describe the social choice model. In Section 3 we define implementation in ex-post equilibria, and we show that all ex-post implementable social choice rules must satisfy PAD. In Section 4 we characterize all ex-post implementable, deterministic social choice rules, subject to two important conditions pertaining to the

\[6\] Although Roberts’ result (which is based on a sophisticated separation argument) is very elegant, it has received, to our knowledge, relatively little attention. We suspect that this is due to the fact that, in the private values framework, there always exist simple and well known mechanisms (the CGV ones) which are both efficient and dominant strategy incentive compatible. Hence there was relatively little interest in mechanisms that were not efficient. This should be contrasted with the interdependent framework where efficient mechanisms usually do not exist.
number of the alternatives and to the signal spaces. We also discuss the role of these conditions. In Section 5 we determine under which conditions the efficient SCR is implementable and relate our results to the results in JM. Section 6 concludes.

2 Model

There is a set \( \mathcal{N} = \{1, \ldots, i, j, \ldots, N\} \) of agents and a set \( \mathcal{K} = \{1, \ldots, k, l, \ldots, K\} \) of alternatives. Each agent \( j \in \mathcal{N} \) has a signal \( s^j \in S^j \) drawn from some topological space. The space of signal combinations of all agents is the Cartesian product \( S := \prod_{j \in \mathcal{N}} S^j \) with generic element \( s \). Whereas \( s^j \) is known only to agent \( j \), it is common knowledge that the signal combination \( s \) is distributed on \( S \) according to some measure with full support.

The space of signal combinations of agents other than \( i \) is denoted \( S^{-i} := \prod_{j \neq i} S^j \), with generic element \( s^{-i} \). As usual, we write \( s = (s^i; s^{-i}) \) when we want to emphasize agent \( i \)'s role.

If the true information is given by \( s \), agent \( i \)'s utility when alternative \( k \) is chosen is given by \( v^i_k(s) + t^i \), where \( v^i_k(s) \) is her valuation for alternative \( k \), and \( t^i \) is a monetary transfer. In addition \( v^i_k(s) \) is given by:

\[
v^i_k(s) = f^i_k(s^i) + h^i_k(s^{-i}),
\]

where \( f^i_k : S^i \rightarrow \mathbb{R} \) and \( h^i_k : S^{-i} \rightarrow \mathbb{R} \) are continuous functions. Writing \( f^i := (f^i_k)_{k \in \mathcal{K}} : S^i \rightarrow \mathbb{R}^K \) we assume that \( f^i(S^i) = \prod_k X^i_k \) is a closed \( K \)-dimensional rectangle, i.e. \( X^i_k \subseteq \mathbb{R} \) is a closed interval for each \( k \in \mathcal{K}. \)

The above framework can accommodate a relatively general model of auctions with allocative and informational externalities. Each alternative describes then how the objects are partitioned among the bidders, and each bidder has a valuation for each partition that depends on the signals obtained by all agents.

A deterministic social choice rule (SCR) chooses one of the \( K \) alternatives, contingent on the signal \( s = (s^i)_{i \in \mathcal{N}} \in S \). In the sequel we always consider deterministic social choice rules (without mentioning it anymore). In addition, we only consider SCRs such that, for each alternative \( k \), there exists at least one signal combination \( s \) where \( k \) is indeed chosen by the SCR.

A SCR \( \psi \) is called efficient, if for every signal \( s \), it maximizes the sum of

\[\begin{align*}
\text{For our results to be meaningful, it is important that } f^i_k(S^i) \text{ has full dimensionality, i.e. a change in } i \text{'s signal can change her relative valuation of the alternatives in every way.}
\end{align*}\]
the agents’ utilities net of monetary transfers:

\[
\psi(s) \in \arg \max_{k \in K} \left\{ \sum_{i=1}^{N} v^i_k(s) \right\}
\]  

(2)

We study here deterministic direct revelation mechanisms\footnote{As usual, a revelation principle applies here.} \( \Gamma = (\psi, t) : S \rightarrow K \times \mathbb{R}^N \), which, for each signal \( s \), specify an alternative \( \psi(s) \in K \) and a payment \( t^i(s) \in \mathbb{R} \) to each agent \( i \).

## 3 Ex-Post Implementation and Positive Association of Differences

Non-trivial dominant strategy equilibria usually do not exist in the presence of interdependent valuations. For this case, the ex-post equilibrium notion can be seen as the analogue to the dominant strategy equilibrium notion in private values models.

**Definition 1** A SCR \( \psi \) is **ex-post implementable** if there are transfers such that truth-telling is an ex-post equilibrium in the game \( G = (S, (\psi, t), u) \):

\[
v^i_{\psi(s)}(s) + t^i(s) \geq v^i_{\psi(s';s^{-i})}(s) + t^i(s';s^{-i})
\]

for every agent \( i \), every true signal \( s = (s^i, s^{-i}) \), and every possible reported signal \( s' : s' \in S^i \).

A crucial role in our analysis will be played by a property of SCRs called ”**positive association of differences**” (PAD), as formulated in Roberts (1979):

**Definition 2** A SCR \( \psi \) satisfies PAD, if and only if for every agent \( i \) and all signals \( s = (s^i, s^{-i}), s' = (s'^i, s'^{-i}) \in S \) such that

\[
v^i_k(s') - v^i_k(s) > v^i_l(s') - v^i_l(s) \quad \text{for all } l \neq k \in K,
\]

\( \psi(s) = k \) implies \( \psi(s') = k \).

PAD is essentially a monotonicity condition: Let the social choice be alternative \( k \) at signal \( s \). Then the choice must remain \( k \) also at signal \( s' \) such that the change from \( s^i \) to \( s'^i \) makes alternative \( k \) relatively more preferable than all other alternatives \( l \neq k \) for an agent \( i \). Note that PAD is an abstract property of SCRs and has, per se, nothing to do with efficiency. This should
be contrasted to single crossing which works by imposing conditions on valuation functions: these conditions induce the necessary alignment between individual and social (i.e., welfare maximizing) incentives. In other words, single crossing indirectly ensures that the efficient SCR satisfies PAD in the respective framework. The importance of PAD is conveyed by the following Lemma:

**Lemma 1** Every ex-post implementable SCR $\psi$ satisfies PAD.

**Proof.** See Appendix. ■

The following simple result shows that the ex-post implementability of an SCR does not depend on how preferences vary in other agents’ signals:

**Lemma 2** An SCR $\psi : S \rightarrow K$ is ex-post implementable in the interdependent values model if and only if it is ex-post implementable in the associated private values model where $\forall i, k, h^i_k \equiv 0$.

**Proof.** See Appendix. ■

### 4 Implementation of General Social Choice Rules

Inspired by the CGV-mechanisms and by the fact that implementability of a given SCR does not depend on the interdependency terms (recall Lemma 2) the following definition introduces weighted variants of the CGV mechanism:

**Definition 3** A SCR $\psi : S \rightarrow K$ is said to be an affine maximizer, if and only if it is of the form:

$$\psi(s) \in \arg \max_{k \in K} \left\{ \sum_{j=1}^{N} \alpha^j f^j_k(s') + \lambda_k \right\}$$

for agent-specific weights $\alpha^j \geq 0$ and alternative-specific weights $\lambda_k \in \mathbb{R}$.

For a given affine maximizer, the weight $\alpha^j$ can be interpreted as the importance of agent $j$’s information to the social choice, and the weight $\lambda_k$ as the designer’s preference for alternative $k$. Note that, for the private values case, a CGV mechanism is an affine maximizer with weights $\alpha^i = 1, \lambda_k = 0$.

When a general SCR is indifferent between two alternatives $k, l$ at some signal combination $s$\(^9\), there is a rate $r_{ij}(s, k, l)$ of information substitution.

\(^9\)This means that $s \in \psi^{-1}\{k\} \cap \psi^{-1}\{l\}$.
between agents $i$ and $j$, such that the SCR stays indifferent if we change $s^i$ and $s^j$ in a way such that agent $i$’s valuation of alternative $k$ and agent $j$’s valuation of alternative $l$ increase at this rate. The main feature of an affine maximizer is that such a rate, which is then given by $\alpha_j/\alpha_i$, depends neither on the signal $s$ nor on the alternatives $k,l$.

We can now state the main result of our paper:

**Theorem 1** 1. Every affine maximizer is ex-post implementable

2. Assume that: 1) $f^i(S^i) = \mathbb{R}^K$ for all $i \in N$ and 2) $K > 2$. Then a SCR $\psi : S \rightarrow \mathcal{K}$ is ex-post implementable only if it is an affine maximizer.

**Proof.** See Appendix. 

Part 1 follows by explicitly constructing the transfer functions $t^i_k$. Part 2 is more subtle and is based on a hyperplane separation argument by Roberts (1979). The proof consists of showing that there is no loss of generality in assuming that an ex-post implementable SCR takes only payoff relevant information into account. Mathematically, this means that $\psi$ *factors* through $f$, i.e. there is $\phi : X \rightarrow \mathcal{K}$ such that $\psi = \phi \circ f$. Using PAD we can then apply Roberts’ Theorem on $\phi$ in order to obtain the characterization in terms of weights.

To better understand what Theorem 1 says, consider the trivial case where only agent $i$ holds private information $s^i$ (while $s^{-i}$ is fixed and known). Agent $i$ has utility $f^i_k(s^i) + h^i_k(s^{-i}) + t^i_i$. Let $\psi$ be implemented by the mechanism $(\psi, t)$. The monetary transfer to agent $i$ may depend on the signal $s^i$ only via the chosen alternative $\psi(s)$, i.e. $t^i_i(s) = t^i_{\psi(s)}(s^{-i})$. Setting $\lambda_k := h^i_k(s^{-i}) + t^i_k(s^{-i})$, the mechanism $(\psi, t)$ induces truthtelling only if $\psi$ is an affine maximizer: $\psi(s) \in \arg\max_{k \in \mathcal{K}} \{f^i_k(s^i) + \lambda_k\}$. For $N > 1$ agents holding private information, the same must be true for any agent $i$ and for any fixed signal $s^{-i}$ of the other agents. Theorem 1 states that the only way to consistently solve the $N$ one-agent problems is via an affine maximizer.

### 4.1 The Role of the Assumptions

The following result characterizes all ex-post implementable SCRs for the case of $K = 2$ alternatives: there are far more implementable SCRs than just affine maximizers. This shows that the requirement $K > 2$ is indispensable for Theorem 1:

**Proposition 1** Assume that $\mathcal{K} = \{k, l\}$ and define reduced signals by $\sigma^i := f^i_k(s^i) - f^i_l(s^i)$. Then a SCR $\psi : S \rightarrow \mathcal{K}$ is ex-post implementable if and only
if there exists an open comprehensive\(^\text{10}\) set \(\Sigma_k \subseteq \mathbb{R}^N\) such that:

\[
\psi(s) = \begin{cases} 
    k & \text{if } (\sigma_1, \ldots, \sigma^N) \in \Sigma_k \\
    l & \text{if } (\sigma_1, \ldots, \sigma^N) \in \mathbb{R}^N \setminus \bar{\Sigma}_k.
\end{cases}
\]

Theorem 3.9. in Laffont, Maskin (1982) is an analogous result for dominant strategy implementation in the private values case. We thus omit the proof which uses techniques found in the proof of Theorem 1 in order to generalize their insight.

The above set \(\Sigma_k\) can be described by a function describing its border. For any agent \(i\) define a function \(g : \mathbb{R}^{N-1} \to \mathbb{R} \cup \{-\infty, \infty\}\) as follows:

\[
g(\sigma^{-i}) := \inf \{\sigma^i : (\sigma^i, \sigma^{-i}) \in \Sigma_k\}.
\]

It is easily seen that \(g\) needs to be weakly monotonic decreasing.

Consider now figure 1 which illustrates the easily graphed case \(N = 2\) (the intuition remains valid for arbitrary \(N\)). The figure shows that the rate of informational substitution \(r_{12}(s, k, k)\) among agents 1 and 2 (which is given by \(-\partial g/\partial \sigma^1\)) needs not be constant. For an affine maximizer, the separating line would be straight with slope \(-\frac{\sigma_2}{\sigma_1}\).

Next we give an example showing that the assumption \(S = (\mathbb{R}^K)^N\) is indispensable in Theorem 1.

\(^{10}\)i.e. a set \(\Sigma_k \subseteq \mathbb{R}^N\) with the property that \(\Sigma_k + \mathbb{R}^N \subseteq \Sigma_k\) (i.e. \(\Sigma_k + \alpha \subseteq \Sigma_k\) for every \(\alpha \geq 0\)).
Example 1 Consider a setting with two agents 1, 2, three alternatives \( \mathcal{K} = \{o, k, l\} \), signal space \( S = ([0, 1]^3)^2 \) and private valuations: \( v_k^i(s) = s_k^i \). Then the social choice rule \( \psi : S \to \mathcal{K} \) defined below is ex-post implementable but it is not an affine maximizer:

\[
\psi(s) = \begin{cases} 
\alpha \max_{i \in \{o, k, l\}} \{-1.5 + s_{o}^1 + s_{o}^2, s_{k}^1 + s_{k}^2, s_{l}^1 + s_{l}^2\} & \text{if } s \in S_{\text{gen}} \\
\psi(o) & \text{if } s \in S \setminus S_{\text{gen}} 
\end{cases}
\]

\[
S_{\text{gen}} = \{s \in S : s_k^1 < s_k^2 + 0.5, \text{ or } s_l^2 > s_k^2 - 0.5\}
\]

**Proof.** Assume by contradiction that \( \psi \) is an affine maximizer with weights \( \alpha^1, \alpha^2, \lambda_o, \lambda_k, \lambda_l \). The formula defining \( \psi \) for \( s \in S_{\text{gen}} \) entails first: \( \alpha^1 = \alpha^2 \) and using this: \( \lambda_k = \lambda_l = \lambda_o + 1.5 \). An affine maximizer with these parameters would choose \( k \) at the signal \( s \) defined by \( s_k^1 = 0.5, s_k^2 = 0.5, s_k^1 = 0.2, s_k^2 = 0.9, s_l^1 = 0.8, s_l^2 = 0.1 \), but as \( s \notin S_{\text{gen}} \) we have \( \psi(s) = l \), a contradiction.

To see that \( \psi \) is ex-post implementable, the reader can check that the transfers

\[
t^1(s) = \begin{cases} 
s_o^2 & \text{if } \psi(s) = o \\
\min \{1.5 + s_k^2, 2 + s_l^2\} & \text{if } \psi(s) = k \\
1.5 + s_l^2 & \text{if } \psi(s) = l
\end{cases}
\]

induce agent 1 to tell the truth. The transfers \( t^2(s) \) can be analogously defined. ■

What goes wrong in Example 1? Consider \( s^2 \) with \( s_k^2 > 0.5 \). No signal \( s^1 \) can lead to \( \psi(s) = o \). Thus, for \( s^2 \) in a subset of \( S^2 \) agent 1 can only ”choose” among alternatives \( k \) and \( l \). For this subset we only get the weak characterization of Proposition 1 instead of the strong one of Theorem 1.

If we severely restrict the class of SCRs in order to rule out the problems of example 1, we get an analogue to Theorem 1.

**Definition 4** Agent \( i \) is decisive for SCR \( \psi \) if, for every signal of others \( s^{-i} \) and for any alternative \( k \), there is a signal \( s^i \) such that \( f^i(s^i) \) is in the interior of \( f^i(S^i) \) and \( \psi(s^i, s^{-i}) = k \).

For example, if \( \psi \) is an affine maximizer, any agent \( j \) with \( \alpha^j > 0 \) and \( f_j^i(S^j) = \mathbb{R}^K \) is decisive for \( \psi \). In particular under the assumptions of Theorem 1, given any ex-post implementable SCR \( \psi \), there is some agent \( i \) who is decisive for \( \psi \).

**Proposition 2** If a SCR \( \psi \) is ex-post implementable and if some agent \( i \) is decisive for \( \psi \), then \( \psi \) is an affine maximizer.

**Proof.** See Appendix. ■
5 Efficient Implementation

The characterization result Theorem 1 implies that the efficient SCR is ex-post implementable only if it is an affine maximizer. This can be interpreted as a necessary condition on the valuation structure. Jehiel and Moldovanu (2001) found an equivalent condition to be necessary for efficient Bayes-Nash implementation in a linear framework with independent signals. In order to relate to their result, we specialize our model to their case.

For this section let $S^i := \prod_{k \in K} S^i_k$ be the cartesian product of $K$ closed intervals and define $f_k^i (s^i) := s_k^i$, $h_k^i (s^{-i}) := \sum_{j \neq i} a_{ki}^j s_k^j$. Valuations are thus given by: $v_k^i (s) = \sum_{j \in N} a_{ki}^j s_k^j$, with $a_{ki}^i = 1$. Thus the coefficient $a_{ki}^j \in \mathbb{R}$ measures the influence of agent $j$’s signal about alternative $k$ on $i$’s valuation of $k$. Note that agent $i$’s signal has an impact of $\sum_{j=1}^N a_{kj}^i$ on social welfare.

A priori, there could be circumstances in which the efficient SCR is implementable in the sense of Bayes-Nash equilibrium, but not in the stronger sense of ex-post equilibrium. The next Proposition shows that the two concepts of efficient implementation coincide in the case of linear valuations\(^{11}\).

**Proposition 3** In the linear setting, the four following statements are equivalent:

1. The efficient SCR is ex-post implementable.
2. The efficient SCR is Bayes-Nash implementable for the case of independent signals.
3. For every agent $i$, the impact of $i$’s signal on social welfare, $\alpha^i = \sum_{j=1}^N a_{kj}^i$, is positive and independent of the alternative $k$.
4. The efficient SCR is an affine maximizer:

$$\psi^\text{eff} (s) = \arg \max_k \left\{ \sum_{j \in N} \alpha^j s_k^j \right\}.$$

**Proof.** $1 \Rightarrow 2$ follows by the definition of the respective implementation concepts; $2 \Rightarrow 3$ is the JM result; $3$ is easily seen to be equivalent to $4$; $4 \Rightarrow 1$ follows by part 1 of Theorem 1. \(\blacksquare\)

\(^{11}\)Note that this is not true in general. There are SCRs that are Bayes-Nash implementable but not ex-post implementable. An example is available from the authors upon request.
The above Proposition relies on the relatively involved proof of Theorem 4.3. in JM.\textsuperscript{12} Remark that \(1 \Rightarrow 3\) could also be obtained by using part 2 of Theorem 1, but this would rely on additional assumptions. We therefore take the opportunity to give a direct and easy proof of the fact that the efficient SCR must be an affine maximizer in order to be ex-post implementable: 

In the linear case, Lemma 1 implies that a change in \(i\)'s signal \(s^i \rightarrow s'^i\) which improves \(i\)'s valuation of \(k\) more than her valuation for \(l\) must also improve the designer’s valuation of \(k\) more than her valuation for \(l\). Letting \(s := (s^i, s^{-i})\), \(s' := (s'^i, s'^{-i})\), this means:

\[
\left( v_{ki}^j (s') - v_{ki}^j (s) \right) - \left( v_{li}^j (s') - v_{li}^j (s) \right) \geq 0 \Rightarrow \\
\left( \sum_{j \in N} v_{kj}^i (s') - v_{kj}^i (s) \right) - \left( \sum_{j \in N} v_{lj}^i (s') - v_{lj}^i (s) \right) \geq 0 \quad (3)
\]

For the linear specification, the above condition holds for all \(s'^i_k - s^i_k, s'^i_l - s^i_l \in \mathbb{R}\) if and only if 

\[
\sum_{j \in N} a_{kj}^i = \sum_{j \in N} a_{lj}^i \geq 0. \quad (*)
\]

In other words, the impact of \(i\)'s signal on social welfare, \(\alpha^i := \sum_{j \in N} a_{kj}^i\), must be positive and independent of the chosen alternative \(k\).

6 Conclusion

Since efficient implementation in settings with interdependent valuations and multidimensional signals is usually not-possible, one has to look for implementable social choice rules that satisfy some other desirable criteria. Given several technical conditions, we have characterized all ex-post incentive compatible SCRs in terms of affine functions of the agents’ signals. We have therefore greatly simplified the quest for other satisfactory mechanisms by reducing the complex design problem to one of determining a finite set of real numbers (representing agent-specific and alternative-specific weights) which yield the desired outcome.

We have assumed that valuations depend in an additively separable manner on own signal and on signals of others. Thus, although an agent cannot

\textsuperscript{12}The main idea there is to check when the conditional expected probability vector field (whose coordinates are the expected probabilities with which an agent with a given signal expects an efficient mechanism to choose the various alternatives) satisfies an integrability constraint imposed by BN incentive compatibility.
infer her valuations from her signal, she knows the marginal valuation with respect to her signal. This is not an innocuous assumption, but, in some sense, it delineates the boundary beyond which the Clarke-Groves-Vickrey insight fails if valuations are interdependent. For arbitrary valuation functions, Jehiel et al (2002) prove a strong impossibility result: generically, an ex-post implementable choice rule cannot take any agent’s information into account, and thus must be constant.

References


**Appendix**

**Proof of Lemma 1.** Let \( \psi \) be ex-post implemented by a mechanism \((\psi, t)\). Consider two signals \( s = (s^i, s^{-i}) \), \( s' = (s'^i, s'^{-i}) \in S \) and denote \( \psi (s) = k \) and \( \psi (s') = k' \). In order to induce truthful revelation of both \( s^i \) and \( s'^i \), it is necessary to have:

\[
\begin{align*}
    f_k^i (s^i) + h_k^i (s^{-i}) + t_k^i (s^{-i}) & \geq f_{k'}^i (s^i) + h_{k'}^i (s^{-i}) + t_{k'}^i (s^{-i}) \\
    f_k^i (s'^i) + h_k^i (s'^{-i}) + t_k^i (s'^{-i}) & \leq f_{k'}^i (s'^i) + h_{k'}^i (s'^{-i}) + t_{k'}^i (s'^{-i})
\end{align*}
\]

Taking differences, we obtain:

\[
    f_k^i (s'^i) - f_k^i (s^i) \leq f_{k'}^i (s'^i) - f_{k'}^i (s^i)
\]

Combining the above inequality with the hypothesis that \( f_k^i (s'^i) - f_k^i (s^i) > f_{k'}^i (s'^i) - f_{k'}^i (s^i) \) for all \( i \in N \) and all \( l \neq k \in K \), yields \( \psi (s') = k' = k \). ■

**Proof of Lemma 2.** "if": Let \( \psi \) be ex-post implemented by a mechanism \((\psi, t)\) under private values. Thus \( f_{\psi(s)}^i (s^i) + t^i (s) = \max_{\hat{s} \in S^i} \left\{ f_{\psi(\hat{s})}^i (s^i) + t^i (\hat{s}) \right\} \) for all \( i, s^i, s^{-i} \), where \( s = (s^i, s^{-i}) \), \( \hat{s} = (\hat{s}^i, s^{-i}) \). For interdependent utilities of the form \( f_k^i (s^i) + h_k^i (s^{-i}) + t^i \) we define monetary payments by \( t^i (s) = t^i (s) - h_{\psi(s)}^i (s^{-i}) \). One easily verifies that \( f_{\psi(s)}^i (s^i) + h_{\psi(s)}^i (s^{-i}) + t^i (s) = \max_{\hat{s} \in S^i} \left\{ f_{\psi(\hat{s})}^i (s^i) + h_{\psi(\hat{s})}^i (s^{-i}) + t^i (\hat{s}) \right\} \) for all \( i, s^i, s^{-i} \). This shows that \((\psi, t')\) ex-post implements \( \psi \) for the interdependent values case.
“only if”: Analogously.

Proof of Theorem 1. 1) Let \( \psi(s) \in \arg \max_{k \in K} \left\{ \sum_{j=1}^{N} \alpha^j f^j_k(s^j) + \lambda_k \right\} \)

Define \( t^i(s) := \frac{1}{\alpha^i} \left( \sum_{j \neq i} \alpha^j f^j_{\psi(s)}(s^j) + \lambda_{\psi(s)} \right) - h^i_{\psi(s)}(s^{-i}) \) for all agents \( i \) such that \( \alpha^i > 0 \). If \( \alpha^i = 0 \), agent \( i \)'s signal is irrelevant for the decision, and we set \( t^i \equiv 0 \).

Given her true signal \( s^i \), and given the other agents’ truthfully reported signals \( s^{-i} \), agent \( i \) faces the problem of what signal \( \tilde{s}^i \) to announce. Denoting \( \tilde{s} := (\tilde{s}^i; s^{-i}) \), her utility is given by:

\[
v^i_{\psi(s)}(s) + t^i(\tilde{s}) = f^i_{\psi(s)}(s) + h^i_{\psi(s)}(s^{-i}) + \frac{1}{\alpha^i} \left( \sum_{j \neq i} \alpha^j f^j_{\psi(s)}(s^j) + \lambda_{\psi(s)} \right) - h^i_{\psi(s)}(s^{-i}) = \frac{1}{\alpha^i} \left( \sum_{j} \alpha^j f^j_{\psi(s)}(s^j) + \lambda_{\psi(s)} \right).
\]

Agent \( i \) optimally chooses \( \tilde{s}^i \) such that \( \psi(s) \) maximizes this expression: \( \tilde{s}^i := s^i \) (here we used \( \alpha^i \geq 0 \)).

2) This part uses an important result due to Roberts (1979). Roberts studied deterministic SCRs that are implementable in dominant strategies in a private values setting, and he showed that such SCRs must satisfy PAD. Using our notation, his proof relies on the following technical result\(^{13}\):

Theorem A (Roberts 1979): Assume that: 1) \( X = (\mathbb{R}^K)^N \) with typical element \( x = ((x^j_k)_{k \in K})_{j \in N} \) and 2) \( K > 2 \). Then any function \( \phi : X \rightarrow K \) which satisfies PAD is an affine maximizer.

Here PAD means that for \( x, x' \in X \) such that

\[
x'^i_k - x^i_k > x'^i_l - x^i_l \quad \text{for all } i \in \mathcal{N} \text{ and all } l \neq k \in \mathcal{K},
\]

\( \phi(s) = k \) implies \( \phi(s') = k \). An affine maximizer is a function \( \phi \) with the property \( \phi(s) \in \arg \max_{k \in K} \left\{ \sum_{j=1}^{N} \alpha^j x^j_k + \lambda_k \right\} \).

Thanks to Lemma 2 we can assume that \( h^i_k \equiv 0 \) for all \( i, k \). In order to apply Theorem A, assume first that \( \psi : S \rightarrow \mathcal{K} \) factors through \( f \), i.e. there is a function \( \phi : X \rightarrow \mathcal{K} \) such that \( \psi = \phi \circ f \). As \( \psi \) is ex-post incentive compatible we can recursively apply the proof of Lemma 1 to show that for all signals \( s, s' \in S \) such that

\[
f^i_k(s') - f^i_k(s) > f^i_l(s') - f^i_l(s) \quad \text{for all } i \in \mathcal{N} \text{ and } l \neq k \in \mathcal{K},
\]

\(^{13}\)Roberts’ proof uses a hyperplane-separation argument which yields the weights in the affine representation.
\(\psi(s) = k\) implies \(\psi(s') = k\). Consider the sequence of signals \(s_{(0)} := s, s_{(i)} := (s'_i, s_{(i-1)})\) for all agents \(i \leq n\) (this gives \(s_{(n)} = s'\)). The proof of Lemma 1 serves then as the induction step proving that with \(\psi(s_{(0)}) = k\) we have \(\psi(s_{(i)}) = k\) for all \(i\), which yields \(\psi(s') = k\).

Thus we can apply Theorem A to \(\phi\) as a function of the \(f_k^i(s)\), and get
\[
\psi(s) = \phi(f(s)) = \arg\max_{k \in K} \left\{ \sum_{j=1}^N \alpha^j f_k^j(s^j) + \lambda_k \right\}.
\]

It remains to show that the above assertion holds also for the cases where \(\psi : S \to K\) does not factor through \(f\). This proof has several steps:  

a) Slightly change \(\psi\) to a function \(\tilde{\psi}\) that factors \(\psi = \phi \circ f\);  
b) show that \(\tilde{\psi}\) is ex-post implementable and apply Theorem A to \(\phi\);  
c) show that if \(\tilde{\psi}\) is an affine maximizer then so is \(\psi\).

a) Given valuation functions \(f^i = (f_k^i)_{k \in K} : S^i \to \mathbb{R}^K\), denote \(f^i(s^i) := x^i\) and for each \(x^i\) fix \(\tilde{s}^i \in (f^i)^{-1}\{x^i\}\). We shall say that \(\tilde{s} = (\tilde{s}^i)_{i \in N}\) represents \(s\). Given an ex-post implementable SCR \(\psi : S \to K\) define \(\tilde{\psi} : S \to K\) by setting
\[
\tilde{\psi}(s) := \psi(\tilde{s})
\]
where \(\tilde{s}\) represents \(s\). Obviously, there is \(\phi : (\mathbb{R}^K)^N \to K\) such that \(\tilde{\psi} = \phi \circ f\).

b) The SCR \(\tilde{\psi}\) is ex-post implementable by the transfer rule \(\tilde{t}((s^i)_{i \in N}) := t((\tilde{s}^i)_{i \in N})\), where \(t : S \to \mathbb{R}^N\) are the transfers that implement \(\psi\). Indeed, we readily check agent \(i\)'s incentive constraint:
\[
\begin{align*}
  f^i_{\psi(s^i, \tilde{s}^i)}(s^i) + \tilde{t}^i(s^i, \tilde{s}^i) &= f^i_{\psi(\tilde{s}^i, \tilde{s}^i)}(s^i) + t^i(\tilde{s}^i, \tilde{s}^i) \geq \\
  f^i_{\psi(\tilde{s}^i, \tilde{s}^i)}(s^i) + t^i(\tilde{s}^i, \tilde{s}^i) &= f^i_{\psi(\tilde{s}^i, \tilde{s}^i)}(s^i) + t^i(s^i, \tilde{s}^i)
\end{align*}
\]
for all \(i, s^i, \tilde{s}^i, s^i, \tilde{s}^i\). The first and third equality follow by the definitions of \(\tilde{\psi}\) and \(t\) and the inequality follows by the ex-post incentive compatibility of \((\psi, t)\). By Lemma 1, \(\psi\) satisfies PAD, which in turn means that \(\phi : (\mathbb{R}^K)^N \to K\) (such that \(\tilde{\psi} = \phi \circ f\)) satisfies PAD in the sense of Roberts' Theorem A. Applying the Theorem, there are constants \(\alpha^j \geq 0\) for \(j \in N\) and \(\lambda_k\) for \(k \in K\) such that \(\phi(f(s)) \in \arg\max_{k \in K} \left\{ \sum_{j=1}^N \alpha^j f_k^j(s^j) + \lambda_k \right\}\) for all \(s \in S\), proving that \(\tilde{\psi}\) is an affine maximizer.

c) We now return to the original SCR \(\psi\). We will derive a contradiction by assuming that there exists \(s \in S\) such that \(\psi(s) = l \notin \arg\max_{k \in K} \left\{ \sum_{j=1}^N \alpha^j f_k^j(s^j) + \lambda_k \right\}\). Consider \(s' \in S\) such that \(f_j^j(s') = f_j^j(s) + \varepsilon\) for all \(j\), \(f_k^j(s') = f_k^j(s)\) for all \(j \neq l\) and \(k \neq l\) with \(\varepsilon\) sufficiently small such that \(l \neq \tilde{\psi}(s') \in \arg\max_{k \in K} \left\{ \sum_{j=1}^N \alpha^j f_k^j(s^j) + \lambda_k \right\}\). Let \(s'\) be the element representing \(s'\) in the definition of \(\tilde{\psi}\). By PAD, \(\psi(s) = l\) implies
ψ(ś′) = l, but, by the characterization of ̃ψ, we know that l ̸= ̃ψ(ś′) contradicting ̃ψ(ś′) = ψ(ś′). This contradiction concludes the proof that ψ(s) ∈ arg max_{k ∈ K} \left\{ \sum_{j=1}^{N} \alpha j f_k^i (s^j) + \lambda_k \right\} for all s ∈ S. ■

**Proof of Proposition 2.** We first show that we can restrict ourselves to SCRs ψ that do not take information into account which does not vary with relative valuations of the alternatives.

We make two assumptions: First, we assume (as in the proof of Theorem 1) that ψ = φ ◦ f factors through f. Second, we choose an (arbitrary) ordering ∋ of the alternatives, and assume that ψ⁻¹ \{l ∈ K : l ∋ k\} is closed for every k ∈ K.14 This allows us to strengthen Lemma 1 as follows:

**Lemma 3** Let ψ be a SCR that is ex-post implementable and satisfies the two assumptions above. Then ψ satisfies PAD', i.e. for signals s, s' ∈ S such that:

\[ f_k^i (s^i_l^0) - f_k^i (s^i_l^0) \geq f_k^i (s^i_l^0) - f_k^i (s^i_l^0) \quad \text{for all } i ∈ N \text{ and all } l \neq k \in K \]

ψ(s) = k implies ψ(s') = k.

**Proof.** Consider s = (s_i, s_j) and s' = (s_i', s_j') as in the statement of the lemma. The proof for general s' follows by induction. In the proof of Lemma 1 we noticed that s_i^k_l^0 - s_i^l_k^0 > s_i^k_l^0 - s_i^l_k^0 ⇒ ψ(s') ≠ l. This entails s_i^k_l^0 - s_i^l_k^0 ≥ s_i^k_l^0 - s_i^l_k^0 ⇒ ψ(s' + εe_k^i) ≠ l 15 for any ε > 0 and therefore ψ(s') ≠ l if k ∋ l. On the other hand, for l ∋ k we have ψ(s + εe_l^k) = k for some ε > 0 yielding ψ(s') ≠ l. We conclude that ψ(s') = k.16 ■ ■

**Proof of Proposition 2 resumed.** Define

\[ \sigma_j^o := π_j^k (x) := x_j^k - x_j^o \text{ for all } k ≠ o \text{ in } K \text{ and all } j \in N \]

where o ∈ K denotes the largest element in K with respect to ∋. Formally, π is a function from X ⊂ (R^K)^N onto Σ ⊂ \prod_{j∈N} Σ^j ⊂ (R^{K-1})^N. We set σ_j^o := 0 for all agents j.

---

14For general ψ we define ̃ψ = φ ◦ f as in the proof of Theorem 1 and φ' by φ'(s) := max \{k ∈ K : x ∈ φ⁻¹ \{k\}\}, where the max is with respect to ∋. Thus ψ' = φ' ◦ f satisfies the assumptions made in the main text, yielding ψ'(s) ∈ arg max_{k ∈ K} \left\{ \sum_{j=1}^{N} \alpha j f_k^i (s^j) + \lambda_k \right\}. This in turn implies ̃ψ(ś) ∈ arg max_{k ∈ K} \left\{ \sum_{j=1}^{N} \alpha j f_k^i (ś) + \lambda_k \right\} (the argument is by contradiction as in part c, of the Proof of Theorem 1.

15ej^k denotes the (j,k)th standard basis vectors in X.

16This proof as stated works only for s, s' ∈ ˇS, the interior of S. For s ∈ S\Š the statement follows by the fact that ψ(s) = max \{k ∈ K : s ∈ ψ⁻¹(k)\}.
The first assumption means that \( \psi \) only takes only payoff relevant information \( f(s) \) into account. The second assumption and Lemma 3 entail that \( \psi \) only depends on the relative valuations \( \sigma = \pi (f(s)) \), rather than absolute valuations implied by the signals \( x = f(s) \). By Lemma 3, \( \psi \) is constant on \( f^{-1}\{\pi^{-1}\{\sigma}\}\) for given \( \sigma \in \Sigma \).\(^{17}\) Thus, there is a choice rule \( \xi : \Sigma \to \mathcal{K} \) that represents \( \psi \) in the reduced variables, i.e. it satisfies

\[
\xi \circ \pi \circ f \equiv \psi.
\]

We now come to the heart of the proof. The idea is to first fix \( s^{-i} \) and describe (as a function of \( s^{i} \)) the choice rules inducing truth telling for agent \( i \). Then we examine the dependence of the choice rule on \( s^{-i} \).

We want to define a function\(^{18}\) \( g : \Sigma^{-i} \to \mathbb{R}^{K-1} \), such that for every signal \( \sigma^{i} \in \Sigma^{i} \):

\[
\phi\left(\sigma^{i}, \sigma^{-i}\right) \in \arg \max_{k \in \mathcal{K}} \left\{ \sigma_{k}^{i} - g_{k}\left(\sigma^{-i}\right) \right\}
\]

The case \( \mathcal{K} = \{o, k, l\} \) is depicted in figure 2. The point \( g(\sigma^{-i}) \) completely describes the choice rule faced by agent \( i \) anticipating \( \sigma^{-i} \).

Our goal is to show that the function \( g_{k} \) is affine of the form:

\[
g_{k}\left(\sigma^{-i}\right) = -\frac{\lambda_{k}}{\alpha^{i}} - \sum_{j \neq i} \frac{\alpha^{j}}{\alpha^{i}} \sigma_{k}^{j}.
\]

This will imply \( \xi(\sigma) \in \arg \max_{k \in \mathcal{K}} \left\{ \sigma_{k}^{i} + \sum_{j \neq i} \frac{\alpha^{j}}{\alpha^{i}} \sigma_{k}^{j} + \frac{\lambda_{k}}{\alpha^{i}} \right\} \), which in turn implies \( \psi(s) = \xi(\pi(f(s))) \in \arg \max_{k \in \mathcal{K}} \left\{ \sum_{j} \alpha^{j} f_{k}^{j}(s^{j}) + \lambda_{k} \right\} \), completing this proof.

By the assumptions that \( i \) is decisive in \( \psi \) and that \( \psi \) induces truth telling of agent \( i \), \( \phi \) must be of the form in equation 4 with \( g_{k}(\sigma^{-i}) = -(t_{k}^{i}(s^{-i}) - t_{o}^{i}(s^{-i})) - (h_{k}^{i}(s^{-i}) - h_{o}^{i}(s^{-i})) \), where \( \pi(s) = \sigma \). This parallels the principal-agent case discussed after Theorem 1.

Lemma 3 yields that \( g_{k} \) is constant with respect to changes in \( \sigma_{j}^{i} \) for all agents \( j \neq i \) and alternatives \( l \neq k \). Suppose not: Then there exist \( \sigma^{-i}, \sigma'^{-i} = \sigma^{-i} + \varepsilon \epsilon_{j}^{i} \)\(^{19}\) for some agent \( j \neq i \) such that \( g_{k}(\sigma^{-i}) > g_{k}(\sigma'^{-i}) \). There is some \( \sigma' \), such that \( \xi(\sigma', \sigma^{-i}) = o \) and \( \xi(\sigma', \sigma'^{-i}) = k \).\(^{20}\) This is in contradiction to Lemma 3, since that Lemma requires \( \xi(\sigma', \sigma'^{-i}) = k \implies \)

\(^{17}\)Note that this is in general not true for \( \psi \) satisfying PAD but not PAD'.

\(^{18}\)The coordinate functions are \( g_{k} \) for alternatives \( k \neq o \). We also set \( g_{o} \equiv 0 \).

\(^{19}\)By slight abuse of notation \( \epsilon_{j}^{i} \) now denotes the \((j, k)\)th standard basis vectors in \( \Sigma \).

\(^{20}\)namely one with \( f_{k}(\sigma^{-i}) > \sigma'^{i} > f_{k}(\sigma'^{-i}) \).
Figure 2: When agent $i$ anticipates $\sigma^{-i}$ (taken fixed in this figure), he "chooses" between $o,k,l$ by announcing $\sigma^i$ in the corresponding area. $i$ is indifferent between the alternatives when his signal $\sigma^i$ is just the indifference point $(f_k(\sigma^{-i}), f_l(\sigma^{-i}))$.

$$\xi(\sigma^i, \sigma^{-i}) = k \text{ if } \sigma^i_j < \sigma^i_j' \text{ and } \xi(\sigma^i, \sigma^{-i}) = o \Rightarrow \xi(\sigma^i, \sigma'^{-i}) = o \text{ if } \sigma^i_j > \sigma^i_j'.$$

Intuitively a change in $\sigma^i_j$ does not affect the relative preferences between $k$ and $o$, and therefore must not "shift the border" between the sets of signals $\sigma^i$ where $\xi$ chooses $k$ and $o$ in figure 2. This entitles us to abuse notation by writing $g_k(\sigma^{-i}) = g_k(\sigma^i_j)$.

An analogous argument, also using Lemma 3, shows that:

$$g_k(\sigma'^{-i}) - g_k(\sigma^{-i}) = g_l(\sigma'^{-i}) - g_l(\sigma^{-i}) \quad (5)$$

for all alternatives $k, l \neq o$ and $\sigma^{-i}, \sigma'^{-i} \in \Sigma^{-i}$ with the property that $\sigma'^{-i}_k - \sigma^{-i}_k = \sigma'^{-i}_l - \sigma^{-i}_l \in \mathbb{R}^{N-1}$. Intuitively, the change from $\sigma^{-i}$ to $\sigma'^{-i}$ does not affect the relative preferences between alternatives $k$ and $l$, and therefore does not "shift the border" between the sets of $\sigma^i$ where $\xi$ chooses $k$ and $l$ in figure 2. Equation 5 entails then

$$g_k(\sigma'^{-i}) - g_k(\sigma^{-i}) = g_k(\sigma'^{-i} + v) - g_k(\sigma^{-i} + v) \quad (6)$$

for all $v \in \mathbb{R}^{N-1}$ such that $\sigma', \sigma, \sigma' + v, \sigma + v \in \Sigma_k^{-i}$. Equation 6 shows that

$$\alpha^j := -\frac{g_k(\sigma'^i + \varepsilon \varepsilon^j) - g_k(\sigma^{-i} + \varepsilon \varepsilon^j)}{\varepsilon} \geq 0 \text{ by Lemma 3}$$

depends neither on $\sigma^{-i}_k$ nor on $\varepsilon^{21}$. It does not depend on $k$ either because of equation 5. Therefore we

$^{21}$The nondependence on $\varepsilon$ follows first for entire and then for rational multiples of a fixed $\varepsilon$ by equation 6, and then for arbitrary multiples by the monotonicity of $f_k$. 

20
have $\frac{\partial g_k}{\partial \sigma_k} \equiv -\alpha^j$ which gives $g_k(\sigma^{-i}) = -\lambda_k - \sum_{j \neq i} \alpha^j \sigma^j_k$ for all $k \neq o$ and all $\sigma^{-i} \in \Sigma^{-i}$. ■