On the Importance of Uniform Sharing Rules for Efficient Matching*

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Abstract

The paper provides a possible explanation for the occurrence of uniform, fixed-proportion rules for sharing surplus in two-sided markets. We study a two-sided matching model with transferable utility where agents are characterized by privately known, multi-dimensional attributes that jointly determine the surplus of each potential partnership. We ask the following question: for what divisions of surplus within matched pairs is it possible to implement the efficient (surplus-maximizing) matching? Our main result shows that the only robust rules compatible with efficient matching are those that divide realized surplus in a fixed proportion, independently of the attributes of the pair's members: each agent must expect to get the same fixed percentage of surplus in every conceivable match. A more permissive result is obtained for one-dimensional attributes and supermodular surplus functions.
Keywords: Matching, surplus division, premuneration values, interdependent values, multi-dimensional attributes.

1 Introduction

The occurrence of uniform, linear rules for sharing surplus among matched agents in a two–sided market - shares that do not vary across matches and are not subject to negotiation - is a widespread and somewhat puzzling phenomenon. For illustration, consider the German law governing the sharing of profit among a public sector employer and an employee arising from the employee's invention activity. Outside universities - where, presumably, the probability of an employee making a job-related discovery is either nil or very low - the law allows any *ex-ante* negotiated contract governing profit sharing (see §40-1 in *Bundesgesetzblatt III, 422-1*). In marked contrast, independently of circumstances, any university and any researcher

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working there must divide the profit from the researcher's invention according to *a fixed 30%-70% rule*, with the employee getting the 30% share (see §42-4).

While the above illustration represented a highly regulated system where the fixed sharing rule is implemented by regulatory fiat, similar arrangements are found in many less regulated environments.[1]

Newbery and Stiglitz (1979) and Allen (1985), among many others, noted that sharecropping contracts in many rural economies involve shares of around one half for landlord and tenant.[2] This division is observed in widely differing circumstances and has persisted for a considerable length of time.[3] The sharecropping literature focused on moral hazard and risk sharing effects - that are absent from our analysis - to explain the continued usage of sharecropping contracts. But, it does not explain the observed uniformity of sharing rules. In this paper, we show that from a mechanism design perspective, uniform, linear sharing rules are important for facilitating efficient matching under incomplete information.

We study a two-sided one-to-one matching (or assignment) market with transferable utility and with a finite number of privately informed agents, called "workers" and "employers." Agents are characterized by multi-dimensional, privately known attributes that jointly determine the value/surplus created by each employer-worker pair. Thus, we discard the prevalent assumption in most incomplete information studies whereby agents can be described by a single trait such as skill, technology, wealth, or education. This is often not tenable, as workers, say, have many diverse job-relevant characteristics, which are only partially correlated.[4]

We take as primitives the agents' utilities from a match in the absence of additional payments - these objects were aptly called "premuneration values" by Mailath, Postlewaite and Samuelson (2012, 2013). These authors also described how premuneration values are shaped by the allocation of property rights: for instance, standardized contracts, as illustrated above, might specify various claims to shares of ex-post realized surplus in every formed partnership. We call the sum of employer and worker premuneration values the match surplus.[5]

We ask the following question: for what forms of premuneration values is there a mechanism that provides incentives for information revelation leading, for each realization of attributes in the economy, to an efficient (surplus-maximizing) matching? We consider standard mechanisms that include payments between agents or to/from a matchmaker at the match formation stage, or, in another interpretation, for mechanisms where agents attempt to signal their attributes and are matched according to these signals.

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[1] In fact, roughly uniform rules for sharing profits from inventive activity are also found across the decentralized university system in the US.

[2] The French and Italian words for "sharecropping" literally mean "50-50 split."

[3] For example, Chao (1983) noted that a fixed 50-50 ratio was prevalent in China for more than 2000 years.

[4] Tinbergen (1956) pioneered the analysis of labor markets where jobs and workers are described by several characteristics. The seminal (complete information) studies of assignment models with traders characterized by multi-dimensional attributes are Shapley and Shubik (1971) and Gretsky, Ostroy and Zame (1992). Dizdar (2012) generalized the matching cum ex-ante investment model due to Cole, Mailath and Postlewaite (2001) along this line, using tools from optimal transport theory. See Villani (2009) for an excellent textbook.

[5] Thus, our model is an incomplete information, interdependent values version of the classical assignment game models of Shapley and Shubik (1971), Crawford and Knoer (1981) and Kelso and Crawford (1982).

Our main result shows that in settings with multi-dimensional, complementary attributes, the only premuneration values compatible with the existence of an incentive compatible, efficient matching mechanism are, in essence,[6] those that correspond to dividing surplus according to a *uniform, fixed proportion*. Thus, to enable efficient match formation, all workers must expect to get the same fixed percentage of surplus in every conceivable match, independently of the attributes of the pair's members, and the same thing must hold for employers! This finding, properly reinterpreted, also has important consequences for markets where only agents on one side - researchers or tenants, say - have private information about their attributes (see Remark 2). Somewhat more flexibility is possible if attributes are one-dimensional and if surplus is supermodular.[7] Efficient matching is then compatible with any division that leaves each partner with a fraction of the surplus that is also supermodular.

An important condition for modeling patterns of complementarity (or substitutability) in assignment problems is the so-called *twist condition*. This is a multi-dimensional generalization of the well-known *strict Spence-Mirrlees condition*, and constitutes the standard assumption about match surplus in the economic literature on (complete information) hedonic equilibrium models (Ekeland, 2010; Chiappori, McCann and Nesheim, 2010) and in the mathematical theory of optimal transport.[8] Moreover, non-singular bilinear match surplus functions, that necessarily satisfy the twist condition, form the most common specification used in empirical studies, following the pioneering contribution of Tinbergen (1956). Chernozhukov, Galichon, Henry and Pass (2014) proved that the twist condition is "just" sufficient for single-market nonparametric identification of hedonic models where agents have several heterogeneous characteristics. As we explain in detail in Section 3, we prove our main result under a condition that is much weaker than the twist condition.

The equilibrium notion used throughout the paper is the *ex-post equilibrium*. This is a generalization of equilibrium in dominant strategies appropriate for settings with interdependent values, and it embodies a notion of no regret: chosen actions must be considered optimal even after the private information of others is revealed. Ex-post equilibrium is a belief-free notion, and our results do not depend in any way on the distribution of attributes in the population.[9]

Our study is at the intersection of several strands of the economic literature. We briefly review below some related papers from each of these strands, emphasizing both the existing relations to our work and the present novel aspects.

1. Matching: An overwhelming majority of studies on two-sided matching has assumed either complete information or *private values*, that is, models in which agents' preferences do not depend on signals privately available to others. In the (private values) Gale-Shapley (1962) model without transfers, one-sided serial dictatorship where women, say, sequentially choose partners according to their preferences leads to a Pareto-optimal matching. Difficul-

[6]See Condition 1, Remark 1 and Theorem 1' for the precise statements.

[7]The complete information version has been popularized by Becker (1973): agents are completely ordered according to their marginal productivity, and efficient matching is assortative.

[8]In this literature - where measures of agents are matched under complete information - the condition is invoked in order to ensure that the optimal transport (the efficient matching) is unique and deterministic (see Villani, 2009).

[9]See also Bergemann and Morris (2005) for the tight connections between ex-post equilibria and "robust design."

ties occur when the stronger *stability* requirement is invoked: a standard result is that no *ex-post stable* matching can be implemented in dominant strategies if both sides of the market are privately informed (see Roth and Sotomayor, 1990).[10] For the case of transferable utility (TU, Shapley and Shubik, 1971) and private values, Yenmez (2013) studied conditions under which weaker stability notions are compatible with ex-post incentive compatibility (and budget feasibility). Similar questions can be addressed for our framework with interdependent values - as in Yenmez's model, ex-post stability is too demanding - but we focus on clarifying when incentive compatibility and surplus-maximization are compatible (which is a trivial question for private values and TU). Liu, Mailath, Postlewaite and Samuelson (2014) developed a notion of incomplete information stability for a matching *that is already in place*, in a TU model with private information about attributes on one side of the market. They showed that the set of incomplete information stable outcomes is a superset of the set of complete information stable outcomes, and they gave sufficient conditions for incomplete information stable matchings to be efficient.

Several recent papers use mechanism design techniques to analyze one-to-one matching problems. In these papers agents have private information about complementary, one-dimensional attributes. Damiano and Li (2007) studied revenue maximization by a designer, given a continuum of agents. They gave conditions for when matching all non-excluded agents assortatively (rather than coarsely, in different meeting places) is optimal. Johnson (2013) studied a similar question for finitely many agents, focussing on indirect implementation through position auctions. Hoppe, Moldovanu and Sela (2009) showed that the efficient, assortative matching can arise in one of the Bayesian equilibria of a bilateral signaling game, i.e. in a setting without designer.[11]

Gomes and Pavan (2015) studied many-to-many matching design by an intermediary and characterized optimal cross subsidization patterns between the two sides of the market. Their agents have two-dimensional types, but their findings are not directly comparable to ours because many-to-many matching and one-to-one matching impose different feasibility requirements, and because the value of a set of partners depends on a cumulative, one-dimensional sufficient statistic in their model. Che, Kim and Kojima (2012) have shown that efficiency is not compatible with ex-post incentive compatibility in a "house-allocation" problem without transfers if agents' values are allowed to depend on information of other agents.

Finally, Mailath, Postlewaite and Samuelson (2012, 2013) studied the role of premuneration values in a model where, before matching, agents make costly investments in their attributes. When personalized pricing - which relies on complete information in the matching market - is not feasible, premuneration values affect investment incentives, and equilibrium investments are generally inefficient.

2. Property Rights: A large literature, following Coase (1960), analyzes the effects of the ex-ante allocation of property rights on bargaining outcomes. The interplay between private information and ex-ante property rights in private value settings has been emphasized

[10] Chakraborty, Citanna and Ostrovsky (2010) showed that there may be no stable matching mechanism even in a one-sided private information model, if preferences on one side of the market (colleges, say) depend on information available to agents on the same side of the market.

[11] All three papers discussed in this paragraph assumed supermodular premuneration values. In Damiano and Li (2007) and Hoppe, Moldovanu and Sela (2009), surplus is also shared fifty-fifty.

by Myerson-Satterthwaite (1983) and Cramton-Gibbons-Klemperer (1987) in a buyer-seller framework and a partnership dissolution model, respectively.[12] In these papers, a value maximizing allocation can be implemented via standard Clarke-Groves-Vickrey schemes. Whenever inefficiencies occur, these stem from the inability to design budget-balanced and individually rational transfers that sustain the value maximizing allocation.[13] In marked contrast, our present analysis completely abstracts from budget-balancedness and individual rationality (with the exception of Remark 4). The fixed-proportion divisions are dictated here by the mere requirement of value maximization together with incentive compatibility.

3. Multi-dimensional Attributes and Mechanism Design: The combination of multi-dimensional attributes, private information and interdependent values is usually detrimental to efficient implementation. In fact, Jehiel, Meyer-ter-Vehn, Moldovanu and Zame (2006) have shown that, generically, only trivial social choice functions - where the outcome does not depend on the agents' private information - can be ex-post implemented when values are interdependent and types are multi-dimensional. Jehiel and Moldovanu (2001) have shown that, generically, the efficient allocation cannot be implemented even if the weaker Bayes-Nash equilibrium concept is used.

Our present insight can be reconciled with those negative results by noting that the assignment game itself has several "non-generic" properties: in particular, the match surplus has the same functional form for every pair (as a function of the respective attributes) and depends neither on how agents outside that pair match, nor on what their attributes are. These features - while natural for the matching model - are non-generic. The sufficiency of fixed-proportion sharing for implementability of the efficient matching is related to the presence of individual utilities that admit a *cardinal* alignment with social welfare, via appropriate Clark-Groves-Vickrey type transfers. By proving necessity, the main result of this paper, we identify an important setting for which efficient implementation is possible *only if* such cardinal alignment is possible.[14] Our result is also reminiscent of Roberts' (1979) characterization of dominant strategy implementation in private values settings, but both technical assumptions and proof are very different here. The analysis of the special case with one-dimensional types and supermodular match surplus is based on an elegant characterization result due to Bergemann and Välimäki (2002), who generalized previous insights due to Jehiel and Moldovanu (2001) and Dasgupta and Maskin (2000).

The paper is organized as follows: in Section 2 we present the matching model. In Section 3 we state our results. Section 4 concludes.

[12] Fieseler, Kittsteiner and Moldovanu (2003) offered a unified treatment that allows for interdependent values and encompasses both the above private values models and Akerlof's (1970) market for lemons.

[13] With several buyers and sellers, the Myerson-Satterthwaite model becomes a one-dimensional, linear incomplete information version of the Shapley-Shubik assignment game. Only in the limit, when the market gets very large, one can reconcile, via almost efficient double-auctions, incentives for information revelation with budget-balancedness and individual rationality. Brusco, Lopomo, Robinson and Viswanathan (2007) and Gärtner and Schmutzler (2009) looked at mergers with interdependent values, a setting which is more related to the present study. However, at most one match is formed in these models, and private information consists of, or can be reduced to, one-dimensional types.

[14] See Section 3 for the link with Jehiel, Meyer-ter-Vehn and Moldovanu's (2008) definition of a *cardinal potential*. These authors presented several non-generic cases where ex-post implementation is possible. See also Bikchandani (2006) for other such cases, e.g. certain auction settings.

2 The Matching Model

There are I employers and J workers. All agents have quasi-linear utilities. Each employer e_i ($i \in \mathcal{I} = \{1, ..., I\}$) privately knows his type $x_i \in X$, and each worker w_j ($j \in \mathcal{J} = \{1, ..., J\}$) privately knows his type $y_j \in Y$. X and Y denote the sets of agents' possible types.

For an employer of type x, the utility from a match with a worker of type y is $\gamma(x, y)v(x, y)$. The worker's utility from such a match is $(1 - \gamma(x, y))v(x, y)$. These *premuneration values* describe utilities in the absence of additional payments. Note that this specification, which takes premuneration values as primitives, follows both the standard mechanism design approach - that specifies the agents' values in terms of the physical allocation - and the classical assignment models.[15] We write here premuneration values as fractions of their sum, the *match surplus* v, to emphasize their dependence on how the gains from partnership formation are divided by pre-specified allocations of property rights or sharing rules.

We assume that the match surplus v satisfies $v : X \times Y \to \mathbb{R}_+$ and that unmatched agents create zero surplus.

Let $\mathcal{M}$ denote the set of all possible one-to-one matchings of employers and workers. If $I \leq J$, these are the injective maps $m : \mathcal{I} \to \mathcal{J}$. A matching $m \in \mathcal{M}$ will be called *efficient* for a type profile $(x_1, ..., x_I, y_1, ..., y_J)$ if and only if it maximizes aggregate surplus

$$u_{m'}(x_1, ..., x_I, y_1, ..., y_J) = \sum_{i=1}^{I} v(x_i, y_{m'(i)})$$

among all $m' \in \mathcal{M}$. Analogous definitions apply for the case $J \leq I$. Efficient matchings are the solutions of a finite linear program (Shapley and Shubik, 1971). We also introduce the notation $v_m^{e_i}$ and $v_m^{w_j}$ for agents' premuneration values in the different matchings $m \in \mathcal{M}$: if e_i and w_j form a match in m, then $v_m^{e_i}(x_1, ..., x_I, y_1, ..., y_J) = \gamma(x_i, y_j)v(x_i, y_j)$ and $v_m^{w_j}(x_1, ..., x_I, y_1, ..., y_J) = (1 - \gamma(x_i, y_j))v(x_i, y_j)$. If e_i is unmatched in m we have $v_m^{e_i}(x_1, ..., x_I, y_1, ..., y_J) = 0$ (similarly, $v_m^{w_j}(x_1, ..., x_I, y_1, ..., y_J) = 0$ if w_j stays unmatched).

This matching model gives rise to a natural social choice setting with interdependent values. Every agent attaches a value to each possible alternative, i.e. matching of employers and workers. This value depends both on the agent's own type and on the type of the partner, but not on the private information of other agents. Moreover, this value does not depend on how other agents match. Thus, there are no allocative externalities, and there are no informational externalities across matched pairs.

2.1 Mechanisms

By the *Revelation Principle*, we may restrict attention to direct revelation mechanisms where truthful reporting by all agents forms an ex-post equilibrium. A direct revelation mechanism (mechanism hereafter) is given by functions $\Psi : X^I \times Y^J \to \mathcal{M}$, $t^{e_i} : X^I \times Y^J \to \mathbb{R}$ and $t^{w_j} : X^I \times Y^J \to \mathbb{R}$, for all $i \in \mathcal{I}$, $j \in \mathcal{J}$.

[15] See Shapley and Shubik (1971), Crawford and Knoer (1981), Kelso and Crawford (1982), Mailath, Postlewaite and Samuelson (2012, 2013), and Liu, Mailath, Postlewaite and Samuelson (2014).

The mechanism Ψ selects a feasible matching as a function of reports, t^{e_i} is the monetary transfer to employer e_i, and t^{w_j} is the monetary transfer to worker w_j, as functions of reports.

Truth-telling is an ex-post equilibrium if for all employers e_i, for all workers w_j, and for all type profiles $p = (x_1, ..., x_I, y_1, ..., y_J)$, $p' = (x_1, ..., x_i', ..., x_I, y_1, ..., y_J)$ and $p'' = (x_1, ..., x_I, y_1, ..., y_j'', ..., y_J)$ it holds that

$$\begin{aligned} v^{e_i}_{\Psi(p)}(p) + t^{e_i}(p) &\geq v^{e_i}_{\Psi(p')}(p) + t^{e_i}(p') \\ v^{w_j}_{\Psi(p)}(p) + t^{w_j}(p) &\geq v^{w_j}_{\Psi(p'')}(p) + t^{w_j}(p''). \end{aligned}$$

3 Results

For which forms of premuneration values, if any, is it possible to implement the surplus-maximizing social choice function in ex-post equilibrium? We start with a simple sufficient condition. All proofs are in the Appendix.

Condition 1. *There is a constant $\lambda_0 \in [0, 1]$ and functions $g : X \to \mathbb{R}$ and $h : Y \to \mathbb{R}$ such that for all $x \in X$, $y \in Y$ it holds that $(\gamma v)(x, y) = \lambda_0 v(x, y) + g(x) + h(y)$. Moreover, h is constant if $I < J$, and g is constant if $I > J$.*

Lemma 1. *If γv satisfies Condition 1, then the efficient matching is implementable in ex-post equilibrium.*

The proof of Lemma 1 is straightforward. Under Condition 1, it is possible to align all agents' utilities with aggregate surplus, via appropriate transfers à la Clark-Groves-Vickrey. When the part of the share that is proportional to match surplus is strictly positive for both sides of the market (i.e. $\lambda_0 \in (0, 1)$), then a *strict* cardinal alignment is possible: in this case, aggregate surplus is a *cardinal potential* for the individual utilities (see Jehiel, Meyer-ter-Vehn and Moldovanu 2008).

Remark 1. *If we require that premuneration values be independent of whether employers or workers are on the short side of the market, then Condition 1 implies that γv is of the form $(\gamma v)(x, y) = \lambda_0 v(x, y) + c$, where $\lambda_0 \in [0, 1]$ and c is a constant. In this case, premuneration values essentially correspond to dividing surplus in the same fixed proportion in all matches (an additional type- and match-independent transfer c is allowed).*

In the remainder of the paper, we assume:

Condition 2. *X and Y are open connected subsets of Euclidean space $\mathbb{R}^n$ for some $n \in \mathbb{N}$, and premuneration values γv and $(1 - \gamma)v$ are continuously differentiable.*

We now turn to our main result: in environments with complementarities or substitutabilities between multi-dimensional types, Condition 1 is necessary for implementing efficient matching. We prove this result under a condition on match surplus that is much weaker - and hence allows for much more general interdependencies - than the twist condition, but we state and explain the twist condition first because the proof is most transparent in this case. We then extend this proof, invoking mainly technical density and continuity arguments, to show the necessity of Condition 1 under the weaker condition.

Condition 3 (Twist)**.** *i) For all $x \in X$, the continuous mapping from Y to $\mathbb{R}^n$ given by $y \mapsto (\nabla_X v)(x, y)$ is injective.*

ii) For all $y \in Y$, the continuous mapping from X to $\mathbb{R}^n$ given by $x \mapsto (\nabla_Y v)(x, y)$ is injective.

Condition 3 implies, in particular, that v is not additively separable with respect to x and y, so that the precise allocation of match partners really matters for efficiency.[16] As a simple example consider the bilinear match surplus $v(x, y) = x \cdot y$, where $\cdot$ denotes the standard inner product on $\mathbb{R}^n$. Then $(\nabla_X v)(x, y) = y$ and $(\nabla_Y v)(x, y) = x$, and Condition 3 is satisfied. More generally, any continuously differentiable surplus function of the form $v(x, y) = a(x) + b(y) + l(x - y)$, where a and b are arbitrary functions and where l is either a strictly concave or strictly convex function, satisfies the twist condition.[17] Note also that for one-dimensional types, the twist condition reduces to the familiar strict super- or submodularity conditions that lead then to the efficient matching being either assortative or anti-assortative: in this case, the twist condition implies that $y \mapsto (\partial_x v)(x, y)$ is either strictly increasing or strictly decreasing. Consequently, v either has strictly increasing differences or strictly decreasing differences in (x, y).[18]

Theorem 1. *Let $n \geq 2$, $I, J \geq 2$, and assume that Conditions 2 and 3 are satisfied. Then the following are equivalent:*

i) The efficient matching is implementable in ex-post equilibrium.

ii) Premuneration values satisfy Condition 1.

Corollary 1. *The only premuneration values for which the efficient matching can be implemented irrespective of whether employers or workers are on the short side of the market are of the form $(\gamma v)(x, y) = \lambda_0 v(x, y) + c$, where $\lambda_0 \in [0, 1]$ and c is a constant.*

The heart of our proof is concerned with situations with two agents on each side (and hence with two feasible matchings), and it exploits the implications of incentive compatibility on the part of employers for varying worker type profiles.

As mentioned earlier, our result is reminiscent of Roberts' (1979) Theorem that shows (under some relatively strong technical conditions) that any dominant-strategy implementable social choice function must maximize a weighted sum of individual utilities plus some alternative-specific constants. Both present assumptions and proof are very different from Roberts'.[19]

Condition 3 encompasses a broad range of interesting complementarities, but its global character reduces its reach. Consider Condition 3i (completely analogous remarks apply for part ii of the condition): for a given (x, y), *local* injectivity of $y' \mapsto (\nabla_X v)(x, y')$ in a neighborhood of y just means that for two different worker types $y' \neq y''$ in the neighborhood, there is some marginal change of x that has different marginal effects on surplus in the

[16] For instance, if v is additively separable and $I = J$, then *all* matchings are efficient, and hence the efficient matching can trivially be implemented, no matter what γ is. This stands in sharp contrast to the result of Theorem 1.

[17] Note that $v(x, y) = x \cdot y$ is obtained by setting $l(z) = -|z|^2/2$ and $a(z) = b(z) = |z|^2/2$.

[18] See also Topkis (1998).

[19] Our main technical result is derived by varying a social choice setting with only two alternatives (Roberts studied a single setting with at least three alternatives), surplus may take here general functional forms, and type spaces are arbitrary connected open sets (Roberts has linear utilities and exploits a "full domain" assumption).

matches with y' and y'', respectively. In particular, if v is twice continuously differentiable this is satisfied if the cross partial $(\nabla^2_{XY}v)(x,y)$ has full rank (for $n=1$, $(\partial^2_{xy}v)(x,y)\neq 0$). What makes Condition 3i restrictive is the requirement of *global* injectivity, which implies in particular that local injectivity must hold *everywhere*. Note also that, in an alternative interpretation, Condition 3i says that for arbitrary $y_1\neq y_2$, the difference $v(\cdot,y_1)-v(\cdot,y_2)$ has no stationary points. The following condition eliminates these global constraints, while retaining the feature that the match surplus exhibits interdependencies that cannot be expressed as a function of lower-dimensional sufficient statistics.

Condition 4. *A) Consider any $y_1\neq y_2$ and $x_1\neq x_2$. Then, there is a smooth curve in X from x_2 to x_1 such that $(\nabla_X v)(x,y_1)-(\nabla_X v)(x,y_2)\neq 0$ a.e. on the curve. Similarly, there is a smooth curve in Y from y_2 to y_1 such that $(\nabla_Y v)(x_1,y)-(\nabla_Y v)(x_2,y)\neq 0$ a.e. on the curve.*[20]

B) For all $x\in X$, the set of $y\in Y$ for which the continuous mapping $y'\mapsto(\nabla_X v)(x,y')$ is not injective on some neighborhood of y has no limit points. For all $y\in Y$, the set of $x\in X$ for which the continuous mapping $x'\mapsto(\nabla_Y v)(x',y)$ is not injective on some neighborhood of x has no limit points.

It may be helpful to illustrate some differences between Condition 3 and Condition 4 in the familiar context of one-dimensional attributes, i.e. for $n=1$. Condition 3 implies that, for any $y_1\neq y_2$, $v(\cdot,y_1)-v(\cdot,y_2)$ is either strictly increasing or strictly decreasing, while Condition 4 allows arbitrarily many alternations in the sign of the derivative of this difference. Similarly, for a twice continuously differentiable surplus, Condition 3 requires that $\partial^2_{xy}v$ does not change signs, while Condition 4 permits $\partial^2_{xy}v$ to change signs arbitrarily many times.

In the Appendix we describe the needed adjustments to the proof of Theorem 1 when we replace Condition 3 by Condition 4. That is, we prove the following strengthening of our main result.

Theorem 1'. *Let $n\geq 2$, $I,J\geq 2$, and assume that Conditions 2 and 4 are satisfied. Then the following are equivalent:*

i) The efficient matching is implementable in ex-post equilibrium.

ii) Premuneration values satisfy Condition 1.

Remark 2 (One-sided Private Information)**.** *Our results have some important implications for settings where only one side of the market has private information. The following claims are immediate consequences of the proofs in the Appendix. Suppose that workers privately know their types while employers' types are common knowledge, and consider the following question under the assumptions of Theorem 1': for what forms of premuneration values is it possible to implement the efficient matching in dominant strategies (employers' types are known, but their premuneration values still matter for efficiency), regardless of the actual profile of employer types, $(x_1,...,x_I)$? The answer is that γv must be of the form $\lambda_0 v(x,y)+g(x)+h(y)$, where $(1-\lambda_0)\in\mathbb{R}_+$, and that h must be constant if $I<J$. An analogous result (with $\lambda_0\in\mathbb{R}_+$ and g constant if $I>J$) applies if only employers privately know their attributes.*

[20] Almost everywhere refers to Lebesgue measure, given the parametrization of the curve.

Remark 3 (Generalized Groves Mechanisms)**.** *Mezzetti (2004) has shown that efficiency is always (that is, in our context, for any given γ) attainable with two-stage "generalized Groves" mechanisms where a final allocation is chosen at stage one, and where, subsequently, monetary transfers that depend on the realized ex-post utilities of all agents at that allocation are executed at stage two.*[21] *In particular, such mechanisms would require ex-post transfers across all existing partnerships, contingent on the previously realized surplus in each of these pairs. We think that using ex-post information (whether reported or verifiable) to this extent is somewhat unrealistic in the present environment. For example, group manipulations by partners should be an issue for any mechanism that imposes ex-post transfers across pairs. In our model, there are no contingent payments between pairs or to/from a potential matchmaker after partnerships have formed.*

Remark 4 (Budget Balance and Individual Rationality)**.** *Assume that the efficient matching can be implemented in ex-post equilibrium (because premuneration values satisfy Condition 1). As usual, if all agents are privately informed, the corresponding mechanism cannot be ex-post budget-balanced (e.g. Laffont and Maskin, 1980, and Yenmez, 2010). However, as in Yenmez (2013), our assumption that match surplus is positive for all (x, y) implies that there is an ex-post incentive compatible, ex-post individually rational mechanism that always creates a budget surplus (the "sidewise Pivot" mechanism). This also implies that for any environment with independently drawn types, there is a* Bayesian *incentive compatible, ex-post budget-balanced and interim individually rational mechanism that always picks the efficient matching. See e.g. Börgers (2015, Ch. 3) for a detailed description of the underlying argument (presented in a public goods context).*

Our second theorem deals with the case where agents' attributes are one-dimensional and v is either strictly supermodular or strictly submodular. This is the classical assortative/anti-assortative framework à la Becker (1973). We treat here the supermodular case. The submodular one is analogous. We find that the class of premuneration values that is compatible with efficient matching is strictly larger than the class defined by Condition 1.

Theorem 2. *Let $n = 1$, $I, J \geq 2$ and assume that Condition 2 holds and that v is strictly supermodular. Then, the efficient matching is implementable in ex-post equilibrium if and only if both γv and $(1 - \gamma)v$ are supermodular.*

We derive Theorem 2 by applying a characterization result due to Bergemann and Välimäki (2002). These authors have provided a necessary as well as a set of sufficient conditions for efficient ex-post implementation for one-dimensional types. The logic of our proof is as follows. We first verify that monotonicity in the sense of Definition 4 of Bergemann and Välimäki is satisfied for strictly supermodular match surplus. This is the first part of their set of sufficient conditions (Proposition 3). Then, we show that their necessary condition (Proposition 1) implies that γv and $(1 - \gamma)v$ must be supermodular. Finally, we show that the second part of the sufficient conditions is satisfied as well if γv and $(1 - \gamma)v$ are supermodular.

[21] The generalized Groves mechanism has the problem that it does not provide *strict* incentives for truthful reporting of ex-post utilities.

4 Conclusion

We have studied a two-sided matching model with a finite number of agents, two-sided incomplete information, interdependent values, and multi-dimensional attributes. We have shown that premuneration values corresponding to uniform, fixed-proportion sharing are essentially the only ones conducive for efficiency in this setting. While our present result is agnostic about the preferred proportion, augmenting our model with, say, a particular ex-ante investment game will introduce new, additional forces that can be used to differentiate between various constant sharing rules. The analysis of relations with incentive compatible core concepts (e.g. Forges, 2004) and various weakenings of ex-post stability (à la Yenmez, 2013) is another interesting direction for further research.

5 Appendix

Proof of Lemma 1. Consider the case $I \leq J$. We make use of the "taxation principle" for ex-post implementation. For employer e_i, and matching $m \in \mathcal{M}$ define $t_m^{e_i}(x_{-i}, y_1, ..., y_J) := \lambda_0 \sum_{l \neq i} v(x_l, y_{m(l)}) - h(y_{m(i)})$. Then, $(\gamma v)(x_i, y_{m(i)}) + t_m^{e_i}(x_{-i}, y_1, ..., y_J) = \lambda_0 \sum_{l=1}^{I} v(x_l, y_{m(l)}) + g(x_i)$, so that it is optimal for e_i to select a matching that maximizes aggregate welfare. Note that strict incentives for truth-telling can be provided only if $\lambda_0 > 0$. For worker w_j, define

$$
\begin{aligned}
t_m^{w_j}(x_1, ..., x_I, y_{-j}) := \quad & (1 - \lambda_0) \sum_{k \in m(\mathcal{I}), k \neq j} v(x_{m^{-1}(k)}, y_k) \\
& + g(x_{m^{-1}(j)}) \mathbf{1}_{j \in m(\mathcal{I})} - h(y_j) \mathbf{1}_{j \notin m(\mathcal{I})}.
\end{aligned}
$$

Here, $\mathbf{1}_{j \in m(\mathcal{I})} = 1$ if $j \in m(\mathcal{I})$, and $\mathbf{1}_{j \in m(\mathcal{I})} = 0$ otherwise. Note that if $I = J$, then $j \in m(\mathcal{I})$ for all possible matchings m, so that the final (y_j-dependent) term always vanishes. If $I < J$, then h is constant by assumption, and the transfer does not depend on y_j. It follows that if w_j is matched in m, his utility is $((1 - \gamma) v)(x_{m^{-1}(j)}, y_j) + t_m^{w_j}(x_1, ..., x_I, y_{-j}) = (1 - \lambda_0) \sum_{k \in m(\mathcal{I})} v(x_{m^{-1}(k)}, y_k) - h(y_j)$. Otherwise, his utility is just $t_m^{w_j}(x_1, ..., x_I, y_{-j}) = (1 - \lambda_0) \sum_{k \in m(\mathcal{I})} v(x_{m^{-1}(k)}, y_k) - h(y_j)$. Hence, it is optimal for w_j to select a matching that maximizes aggregate welfare (strict incentives for truth-telling can be provided only if $\lambda_0 < 1$). This proves the claim for $I \leq J$. The proof for the case $I \geq J$ is completely analogous. $\square$

We prepare the proof of Theorem 1 by a sequence of lemmas. The key step is Lemma 5 below. It will be very useful to introduce a cross-difference (two-cycle) linear operator F, which acts on functions $f : X \times Y \to \mathbb{R}$. The operator F_f has arguments $x^1 \in X^1 = X$, $x^2 \in X^2 = X$, $y^1 \in Y^1 = Y$ and $y^2 \in Y^2 = Y$, and it is defined as follows:[22]

$$
\begin{aligned}
F_f(x^1, x^2, y^1, y^2) &:= f(x^1, y^1) + f(x^2, y^2) - f(x^1, y^2) - f(x^2, y^1) \\
&= (f(x^1, y^1) - f(x^1, y^2)) - (f(x^2, y^1) - f(x^2, y^2)).
\end{aligned}
$$

[22]We use superscripts here because x_1 is already reserved for the type of employer e_1, and so on.

We also define the sets

$$A := \{(x^1, x^2, y^1, y^2) \in X \times X \times Y \times Y | F_v(x^1, x^2, y^1, y^2) = 0\},$$

and

$$A_0 := \{(x^1, x^2, y^1, y^2) \in A | \nabla F_v(x^1, x^2, y^1, y^2) \neq 0\},$$

where $\nabla F_v(x^1, x^2, y^1, y^2) = (\nabla_{X^1} F_v, \nabla_{X^2} F_v, \nabla_{Y^1} F_v, \nabla_{Y^2} F_v)(x^1, x^2, y^1, y^2)$.

Whenever $x_1 \neq x_2$ or $y_1 \neq y_2$, Condition 3 implies that $\nabla F_v(x_1, x_2, y_1, y_2) \neq 0$. This is repeatedly used below.

Lemma 2. *Let $n \in \mathbb{N}$, $I = J = 2$, and let Conditions 2 and 3 be satisfied. If the efficient matching is ex-post implementable, then the following implications hold for all (x_1, x_2, y_1, y_2):*

$$F_v(x_1, x_2, y_1, y_2) > (<)\, 0 \Rightarrow F_{\gamma v}(x_1, x_2, y_1, y_2) \geq (\leq)\, 0, \tag{1}$$

$$F_v(x_1, x_2, y_1, y_2) > (<)\, 0 \Rightarrow F_{(1-\gamma)v}(x_1, x_2, y_1, y_2) \geq (\leq)\, 0. \tag{2}$$

Proof of Lemma 2. There are only two alternative matchings, $m_1 = ((e_1, w_1), (e_2, w_2))$ and $m_2 = ((e_1, w_2), (e_2, w_1))$. Since the efficient matching is ex-post implementable, the taxation principle for ex-post implementation implies that there must be "transfer" functions $t^{e_1}_{m_1}(x_2, y_1, y_2)$ and $t^{e_1}_{m_2}(x_2, y_1, y_2)$ for employer e_1 such that

$$\begin{aligned} F_v(x_1, x_2, y_1, y_2) &> (<)\, 0 \Rightarrow \\ (\gamma v)(x_1, y_1) + t^{e_1}_{m_1}(x_2, y_1, y_2) &\geq (\leq)\, (\gamma v)(x_1, y_2) + t^{e_1}_{m_2}(x_2, y_1, y_2). \end{aligned} \tag{3}$$

For $y_1 \neq y_2$, we have $(\nabla_{X^1} F_v)(x_2, x_2, y_1, y_2) = (\nabla_X v)(x_2, y_1) - (\nabla_X v)(x_2, y_2) \neq 0$ by Condition 3. Hence, in every neighborhood of $x_1 = x_2$, there are x_1' and x_1'' such that $F_v(x_1', x_2, y_1, y_2) > 0$ and $F_v(x_1'', x_2, y_1, y_2) < 0$. Since γv is continuous, relation (3) then pins down the difference of transfers as:

$$t^{e_1}_{m_1}(x_2, y_1, y_2) - t^{e_1}_{m_2}(x_2, y_1, y_2) = (\gamma v)(x_2, y_2) - (\gamma v)(x_2, y_1).$$

Plugging this back into (3) yields (1) for all (x_1, x_2, y_1, y_2). A completely analogous argument based on the incentive constraints for worker w_1 yields (2). □

To prove Theorem 1, we only need local versions of (1) and (2) at profiles where the efficient matching changes. These are available for general $I, J \geq 2$:

Lemma 3. *Let $n \in \mathbb{N}$, $I, J \geq 2$ and let Conditions 2 and 3 be satisfied. If the efficient matching is ex-post implementable, then for all $(x_1, x_2, y_1, y_2) \in A$, there is an open neighborhood $U_{(x_1,x_2,y_1,y_2)} \subset X \times X \times Y \times Y$ of (x_1, x_2, y_1, y_2) such that for all $(x_1', x_2', y_1', y_2') \in U_{(x_1,x_2,y_1,y_2)}$:*

$$F_v(x_1', x_2', y_1', y_2') > (<)\, 0 \Rightarrow F_{\gamma v}(x_1', x_2', y_1', y_2') \geq (\leq)\, 0, \tag{4}$$

$$F_v(x_1', x_2', y_1', y_2') > (<)\, 0 \Rightarrow F_{(1-\gamma)v}(x_1', x_2', y_1', y_2') \geq (\leq)\, 0. \tag{5}$$

Proof of Lemma 3. Given $(x_1, x_2, y_1, y_2) \in A$, fix the types of all other employers and workers (x_i for $i \neq 1, 2$, y_j for $j \neq 1, 2$) such that there is an open neighborhood $U_{(x_1,x_2,y_1,y_2)}$ of (x_1, x_2, y_1, y_2) with the following property: for all $(x_1', x_2', y_1', y_2') \in U_{(x_1,x_2,y_1,y_2)}$, the efficient matching for the profile $(x_1', x_2', x_3, ..., x_I, y_1', y_2', y_3, ..., y_J)$ either matches e_1 to w_1 and e_2 to w_2, or e_1 to w_2 and e_2 to w_1 (depending on the sign of $F_v(x_1', x_2', y_1', y_2')$). From here on, the proof parallels the one of Lemma 2. □

Lemma 3 has the immediate consequence that on A_0, the gradients of F_v, $F_{\gamma v}$ and $F_{(1-\gamma)v}$ must all point in the same direction:

Lemma 4. *Let $n \in \mathbb{N}$, $I, J \geq 2$ and let Conditions 2 and 3 be satisfied. If the efficient matching is ex-post implementable, then there is a unique function $\lambda : A_0 \to [0, 1]$ satisfying*

$$\nabla F_{\gamma v}(x_1, x_2, y_1, y_2) = \lambda(x_1, x_2, y_1, y_2)\nabla F_v(x_1, x_2, y_1, y_2) \tag{6}$$

for all $(x_1, x_2, y_1, y_2) \in A_0$.

Proof of Lemma 4. For any $(x_1, x_2, y_1, y_2) \in A_0$, $\nabla F_v(x_1, x_2, y_1, y_2) \neq 0$. Thus, A is locally a differentiable manifold of codimension 1, and (4) then implies that there is a unique $\lambda(x_1, x_2, y_1, y_2) \geq 0$ such that

$$\nabla F_{\gamma v}(x_1, x_2, y_1, y_2) = \lambda(x_1, x_2, y_1, y_2)\nabla F_v(x_1, x_2, y_1, y_2).$$

Moreover, $\nabla F_{(1-\gamma)v}(x_1, x_2, y_1, y_2) = (1-\lambda(x_1, x_2, y_1, y_2))\nabla F_v(x_1, x_2, y_1, y_2)$ and (5) therefore implies $\lambda(x_1, x_2, y_1, y_2) \in [0, 1]$. □

The crucial step in the proof follows now. It shows that for $n \geq 2$ the function λ must be constant. This constant corresponds then to a particular fixed-proportion sharing rule.

Lemma 5. *Let $n \geq 2$, $I, J \geq 2$ and let Conditions 2 and 3 be satisfied. Then the function λ from Lemma 4 must be constant: there is a $\lambda_0 \in [0, 1]$ such that $\lambda \equiv \lambda_0$.*

Proof of Lemma 5. Let us spell out the equalities in (6):

$$\begin{aligned}
(\nabla_X \gamma v)(x_1, y_1) - (\nabla_X \gamma v)(x_1, y_2) &= \lambda(x_1, x_2, y_1, y_2)((\nabla_X v)(x_1, y_1) - (\nabla_X v)(x_1, y_2)) \\
(\nabla_X \gamma v)(x_2, y_2) - (\nabla_X \gamma v)(x_2, y_1) &= \lambda(x_1, x_2, y_1, y_2)((\nabla_X v)(x_2, y_2) - (\nabla_X v)(x_2, y_1)) \\
(\nabla_Y \gamma v)(x_1, y_1) - (\nabla_Y \gamma v)(x_2, y_1) &= \lambda(x_1, x_2, y_1, y_2)((\nabla_Y v)(x_1, y_1) - (\nabla_Y v)(x_2, y_1)) \\
(\nabla_Y \gamma v)(x_2, y_2) - (\nabla_Y \gamma v)(x_1, y_2) &= \lambda(x_1, x_2, y_1, y_2)((\nabla_Y v)(x_2, y_2) - (\nabla_Y v)(x_1, y_2)).
\end{aligned} \tag{7}$$

Given any $(x_1, x_2, y_1, y_2) \in A_0$, one obtains the same system of equations at $(x_2, x_1, y_1, y_2) \in A_0$, albeit for $\lambda(x_2, x_1, y_1, y_2)$. Thus, the function λ is symmetric with respect to x_1 and x_2. Similarly, it is symmetric with respect to y_1 and y_2. Next, for given $x_1 \in X$ and $y_1 \neq y_2$, the vectors in the first equation of (7) (with $(\nabla_X v)(x_1, y_1) - (\nabla_X v)(x_1, y_2) \neq 0$ on the right hand side) do not depend on how (x_1, y_1, y_2) is completed by x_2 to yield a full profile that lies in A_0. Consequently, $\lambda(x_1, x_2, y_1, y_2) = \lambda(x_1, x_1, y_1, y_2)$ for all these possible choices.

We next show that for a given x_1, λ does in fact not depend on y_1 and y_2 as long as $y_1 \neq y_2$. To this end, start with any $x_1 \in X$ and $y_1 \neq y_2$. We will show that for all $y_2' \neq y_1$,

$$\lambda(x_1, x_1, y_1, y_2) = \lambda(x_1, x_1, y_1, y_2'). \tag{8}$$

Then, by symmetry of λ, $\lambda(x_1, x_1, y_1, y_2) = \lambda(x_1, x_1, y_2', y_1)$, and repeating the argument implies that $\lambda(x_1, x_1, \cdot, \cdot)$ is indeed constant.

So, let us prove (8). Using the first equation of (7), we have:

$$\begin{aligned}
&\lambda(x_1, x_1, y_1, y_2)((\nabla_X v)(x_1, y_1) - (\nabla_X v)(x_1, y_2)) \\
&\quad = ((\nabla_X \gamma v)(x_1, y_1) - (\nabla_X \gamma v)(x_1, y_2')) + ((\nabla_X \gamma v)(x_1, y_2') - (\nabla_X \gamma v)(x_1, y_2)) \\
&\quad = \lambda(x_1, x_1, y_1, y_2')((\nabla_X v)(x_1, y_1) - (\nabla_X v)(x_1, y_2')) \\
&\quad + \lambda(x_1, x_1, y_2', y_2)((\nabla_X v)(x_1, y_2') - (\nabla_X v)(x_1, y_2)).
\end{aligned}$$

It follows that

$$\begin{aligned}
&(\lambda(x_1, x_1, y_1, y_2') - \lambda(x_1, x_1, y_1, y_2))\,((\nabla_X v)(x_1, y_1) - (\nabla_X v)(x_1, y_2')) \\
&\quad + (\lambda(x_1, x_1, y_2', y_2) - \lambda(x_1, x_1, y_1, y_2))\,((\nabla_X v)(x_1, y_2') - (\nabla_X v)(x_1, y_2)) \\
&\quad = 0.
\end{aligned} \tag{9}$$

Two cases must now be distinguished.

Case 1: $(\nabla_X v)(x_1, y_1) - (\nabla_X v)(x_1, y_2')$ and $(\nabla_X v)(x_1, y_2') - (\nabla_X v)(x_1, y_2)$ are linearly independent. Then, it follows from (9) that $\lambda(x_1, x_1, y_1, y_2') = \lambda(x_1, x_1, y_1, y_2)$.

Case 2: $(\nabla_X v)(x_1, y_1) - (\nabla_X v)(x_1, y_2')$ and $(\nabla_X v)(x_1, y_2') - (\nabla_X v)(x_1, y_2)$ are linearly dependent. In this case, pick some $y_2'' \in Y$ such that $(\nabla_X v)(x_1, y_1) - (\nabla_X v)(x_1, y_2'')$ and $(\nabla_X v)(x_1, y_2'') - (\nabla_X v)(x_1, y_2)$ are linearly independent. This is always possible as $(\nabla_X v)(x_1, \cdot)$ maps open neighborhoods of y_1 one-to-one into $\mathbb{R}^n$, and since for $n \geq 2$, there is no one-to-one continuous mapping from an open set in $\mathbb{R}^n$ to the real line $\mathbb{R}$.[23]

From Case 1, we obtain $\lambda(x_1, x_1, y_1, y_2'') = \lambda(x_1, x_1, y_1, y_2)$. As $(\nabla_X v)(x_1, y_1) - (\nabla_X v)(x_1, y_2')$ and $(\nabla_X v)(x_1, y_2') - (\nabla_X v)(x_1, y_2'')$ are also linearly independent, we then get $\lambda(x_1, x_1, y_1, y_2') = \lambda(x_1, x_1, y_1, y_2'')$, and hence (8) follows.

The third equation of (7) may be now used in an analogous way to show that for a given y_1, $\lambda(x_1, x_2, y_1, y_1)$ does not depend on x_1 and x_2, as long as $x_1 \neq x_2$.

The final ingredient is the following observation: for every $(x_1, x_1, y_1, y_2) \in A_0$, there is a $x_2 \neq x_1$ with $(x_1, x_2, y_1, y_2) \in A_0$. Indeed, $(\nabla_{X^2} F_v)(x_1, x_1, y_1, y_2) \neq 0$, so that the set of x_2 for which $(x_1, x_2, y_1, y_2) \in A_0$ is given locally (in a neighborhood of $x_2 = x_1$) by a differentiable manifold of dimension $n - 1$. Since $n \geq 2$, this manifold must contain points other than x_1. A similar argument applies to $(x_1, x_2, y_1, y_1) \in A_0$.

To conclude the proof, we show that λ is constant on $\{(x_1, x_2, y_1, y_2) \in A_0 | x_1 \neq x_2 \text{ and } y_1 \neq y_2\}$. This set is non-empty by the previous observation (and we have already seen that $\lambda(x_1, x_2, y_1, y_2) = \lambda(x_1, x_1, y_1, y_2)$ and $\lambda(x_1, x_2, y_1, y_2) = \lambda(x_1, x_2, y_1, y_1)$, so that λ is constant on all of A_0 then). Given any $(x_1, x_2, y_1, y_2), (x_1', x_2', y_1', y_2') \in A_0$ with $x_1 \neq x_2$, $y_1 \neq y_2$,

[23] This is a special case of Brouwer's (1911) classical dimension preservation result: For $k < m$, there is no one-to-one, continuous function from a non-empty open set U of $\mathbb{R}^m$ into $\mathbb{R}^k$.

$x_1' \neq x_2'$ and $y_1' \neq y_2'$, we have:

$$\begin{aligned}&\lambda(x_1, x_2, y_1, y_2) = \lambda(x_1, x_1, y_1, y_2) = \lambda(x_1, x_1, y_1', y_2') \\ = \;& \lambda(x_1, x_2'', y_1', y_2') = \lambda(x_1, x_2'', y_1', y_1') = \lambda(x_1', x_2', y_1', y_1') = \lambda(x_1', x_2', y_1', y_2'),\end{aligned}$$

where $x_2'' \neq x_1$ is any feasible profile completion for (x_1, y_1', y_2'). □

We are now ready to prove Theorem 1.

Proof of Theorem 1. **ii)⇒ i):** See Lemma 1.

i)⇒ ii): By Lemma 5, there is a $\lambda_0 \in [0,1]$ such that for all $x \in X$, $y_1, y_2 \in Y$ with $y_1 \neq y_2$ it holds (the profile may be completed to lie in A_0, e.g. by $x' = x$):

$$(\nabla_X \gamma v)(x, y_1) - (\nabla_X \gamma v)(x, y_2) \;=\; \lambda_0((\nabla_X v)(x, y_1) - (\nabla_X v)(x, y_2)).$$

Integrating along any path from x_2 to x_1 (X is open and connected in $\mathbb{R}^n$, hence path-connected) yields $F_{\gamma v}(x_1, x_2, y_1, y_2) = \lambda_0 F_v(x_1, x_2, y_1, y_2)$. Hence, by linearity of the operator F, we obtain that $F_{(\gamma-\lambda_0)v} \equiv 0$. A function of two variables has vanishing cross differences if and only if it is additively separable, so that we can write $(\gamma v)(x, y) = \lambda_0 v(x, y) + g(x) + h(y)$. This concludes the proof for the case where $I = J$.

It remains to prove that h must be constant if $I < J$ (the proof that g must be constant when $I > J$ is analogous). Given $y_1 \in Y$, Condition 3 implies that $(\nabla_Y v)(\cdot, y_1)$ vanishes at most in one point. Pick then any $x_1 \in X$ with $(\nabla_Y v)(x_1, y_1) \neq 0$. Set $y_2 = y_1$ and complete the type profile for $(i \neq 1,\ j \neq 1, 2)$ such that, for an open neighborhood U of (y_1, y_1), the efficient matching changes only with respect to the partner of e_1 : either w_1 is matched to e_1 and w_2 remains unmatched, or w_2 is matched to e_1 and w_1 remains unmatched. For $(y_1', y_2') \in U$, it follows that $v(x_1, y_1') - v(x_1, y_2') \geq (\leq)\ 0$ implies $((1-\gamma)v)(x_1, y_1') - ((1-\gamma)v)(x_1, y_2') \geq (\leq)\ 0$. Hence, there is a $\mu(x_1, y_1) \geq 0$ such that

$$(1 - \lambda_0)(\nabla_Y v)(x_1, y_1) - (\nabla_Y h)(y_1) = \mu(x_1, y_1)(\nabla_Y v)(x_1, y_1).$$

In other words, $(\nabla_Y h)(y_1)$ and $(\nabla_Y v)(x_1, y_1)$ are linearly dependent. Finally, let x_1 vary and note that, by Condition 3, the image of $(\nabla_Y v)(\cdot, y_1)$ cannot be concentrated on a line (recall footnote 20). Thus, we obtain that $(\nabla_Y h)(y_1) = 0$. Since y_1 was arbitrary and Y is connected, it follows that the function h must constant. □

Proof of Theorem 1'. First, we show that the result of Lemma 4 continues to hold if Condition 3 is replaced by Condition 4A. Then, we detail the necessary changes to the proofs of Lemma 5 and Theorem 1 if Condition 3 is replaced by Condition 4.

Lemma 6 (Generalizing Lemma 4)**.** *Let $n \in \mathbb{N}$, $I, J \geq 2$ and let Condition 2 and Condition 4A be satisfied. If the efficient matching is ex-post implementable, then there is a unique function $\lambda : A_0 \to [0,1]$ satisfying*

$$\nabla F_{\gamma v}(x_1, x_2, y_1, y_2) = \lambda(x_1, x_2, y_1, y_2) \nabla F_v(x_1, x_2, y_1, y_2)$$

for all $(x_1, x_2, y_1, y_2) \in A_0$.

Proof of Lemma 6. We show that (1) and (2) hold for all (x_1, x_2, y_1, y_2). The proofs of Lemmas 3 and 4 then apply without any changes.

Fix $y_1 \neq y_2$ and an arbitrary x_2. By the taxation principle, there are transfers $t^{e_1}_{m_1}(x_2, y_1, y_2)$ and $t^{e_1}_{m_2}(x_2, y_1, y_2)$ such that for for all $x_1 \in X$ implication (3) holds. If x_2 is not a local extremum of $v(\cdot, y_1) - v(\cdot, y_2)$, then in every neighbourhood of $x_1 = x_2$, there exist x_1' and x_1'' such that $F_v(x_1', x_2, y_1, y_2) > 0$ and $F_v(x_1'', x_2, y_1, y_2) < 0$. As in the proof of Lemma 2, (3) and continuity of γv pin down $t^{e_1}_{m_1}(x_2, y_1, y_2) - t^{e_1}_{m_2}(x_2, y_1, y_2)$, so that implication (1) holds for the given (x_2, y_1, y_2) and arbitrary x_1. An analogous argument for employer e_2 shows that (1) holds if $y_1 \neq y_2$, x_1 is not a local extremum of $v(\cdot, y_1) - v(\cdot, y_2)$, and x_2 is arbitrary. In sum, (1) holds for all (x_1, x_2, y_1, y_2), such that $y_1 \neq y_2$ and at least one of x_1 and x_2 is not a local extremum of $v(\cdot, y_1) - v(\cdot, y_2)$.

Consider now an arbitrary (x_1, x_2, y_1, y_2) satisfying $F_v(x_1, x_2, y_1, y_2) > 0$. By the first part of Condition 4A, there exists a sequence $(x_1^k)_{k\in\mathbb{N}}$ with $\lim_{k\to\infty} x_1^k = x_1$ such that for all k, x_1^k is not a local extremum of $v(\cdot, y_1) - v(\cdot, y_2)$. By continuity of F_v, there is a $K \in \mathbb{N}$ such that for all $k \geq K$, $F_v(x_1^k, x_2, y_1, y_2) > 0$, implying $F_{\gamma v}(x_1^k, x_2, y_1, y_2) \geq 0$. Taking the limit and using the continuity of $F_{\gamma v}$ shows that $F_v(x_1, x_2, y_1, y_2) > 0$ indeed implies $F_{\gamma v}(x_1, x_2, y_1, y_2) \geq 0$. $F_v(x_1, x_2, y_1, y_2) < 0 \Rightarrow F_{\gamma v}(x_1, x_2, y_1, y_2) \leq 0$ follows in the same way. This proves (1).

A completely analogous argument, using the incentive constraints for w_1 and w_2 and the second part of Condition 4A shows that (2) holds for all (x_1, x_2, y_1, y_2). □

Our generalization of Lemma 5 involves the following subset of A_0:

$$A_0^R := \{(x_1, x_2, y_1, y_2) \in A |\, \text{if } y_1 \neq y_2 \text{ then } (\nabla_{(X^1,X^2)} F_v)(x_1, x_2, y_1, y_2) \neq 0, \\ \text{if } x_1 \neq x_2 \text{ then } (\nabla_{(Y^1,Y^2)} F_v)(x_1, x_2, y_1, y_2) \neq 0\}.$$

That is, if $y_1 \neq y_2$, at least one of x_1 and x_2 is not a stationary point of $v(\cdot, y_1) - v(\cdot, y_2)$, and if $x_1 \neq x_2$, at least one of y_1 and y_2 is not a stationary point of $v(x_1, \cdot) - v(x_2, \cdot)$.

Lemma 7 (Generalizing Lemma 5)**.** *Let $n \geq 2$, $I, J \geq 2$ and let Conditions 2 and 4 be satisfied. Then the function λ from Lemma 6 is constant on A_0^R: there is a $\lambda_0 \in [0, 1]$ such that $\lambda \equiv \lambda_0$ on A_0^R.*

Proof of Lemma 7. The symmetries $\lambda(x_1, x_2, y_1, y_2) = \lambda(x_2, x_1, y_1, y_2) = \lambda(x_1, x_2, y_2, y_1)$ and the property that for all $(x_1, x_1, y_1, y_2), (x_1, x_2, y_1, y_2) \in A_0^R$, $\lambda(x_1, x_2, y_1, y_2) = \lambda(x_1, x_1, y_1, y_2)$ follow exactly as in the proof of Lemma 5.

Next, we extend the crucial step of Lemma 5: for all (x_1, x_1, y_1, y_2),$(x_1, x_1, y_1, y_2') \in A_0^R$, $\lambda(x_1, x_1, y_1, y_2) = \lambda(x_1, x_1, y_1, y_2')$, that is, the analog of (8) holds. Applying this twice yields $\lambda(x_1, x_1, y_1, y_2) = \lambda(x_1, x_1, y_1', y_2')$ for all (x_1, x_1, y_1, y_2), $(x_1, x_1, y_1', y_2') \in A_0^R$.[24]

There are two possibilities to consider. If $(\nabla_X v)(x_1, y_2') - (\nabla_X v)(x_1, y_2) \neq 0$, the argument in the proof of Lemma 5 applies. The only change is that we now invoke the first part of Condition 4B, which implies that the image of Y under $(\nabla_X v)(x_1, \cdot)$ is not contained in straight line, to ensure the existence of y_2'' in Case 2. If $(\nabla_X v)(x_1, y_2') -$

[24]Indeed, let $\{y_1, y_2\} \cap \{y_1', y_2'\} = \emptyset$. If $(x_1, x_1, y_1, y_1') \in A_0^R$ then $\lambda(x_1, x_1, y_1, y_2) = \lambda(x_1, x_1, y_1, y_1') = \lambda(x_1, x_1, y_1', y_1) = \lambda(x_1, x_1, y_1', y_2')$. If $(x_1, x_1, y_1, y_1') \notin A_0^R$, then $(x_1, x_1, y_1, y_2') \in A_0^R$ (as $(\nabla_X v)(x_1, y_1) - (\nabla_X v)(x_1, y_2') = (\nabla_X v)(x_1, y_1') - (\nabla_X v)(x_1, y_2') \neq 0$ by assumption then), so that $\lambda(x_1, x_1, y_1, y_2) = \lambda(x_1, x_1, y_1, y_2') = \lambda(x_1, x_1, y_1', y_2')$.

$(\nabla_X v)(x_1, y_2) = 0$, then by Condition 4A every neighbourhood of x_1 contains some $\hat{x}_1$ such that $(\nabla_X v)(\hat{x}_1, y_2') - (\nabla_X v)(\hat{x}_1, y_2) \neq 0$. As $(x_1, x_1, y_1, y_2), (x_1, x_1, y_1, y_2') \in A_0^R$, we also have $(\hat{x}_1, \hat{x}_1, y_1, y_2), (\hat{x}_1, \hat{x}_1, y_1, y_2') \in A_0^R$ for $\hat{x}_1$ sufficiently close to x_1, by continuity of $\nabla_X v$. Hence, by the argument just given for the first possibility, we have $\lambda(\hat{x}_1, \hat{x}_1, y_1, y_2) = \lambda(\hat{x}_1, \hat{x}_1, y_1, y_2')$ for all these perturbations $\hat{x}_1$. $\lambda(x_1, x_1, y_1, y_2) = \lambda(x_1, x_1, y_1, y_2')$ then follows from continuity of $\lambda(\cdot)$. An analogous argument yields $\lambda(x_1, x_2, y_1, y_1) = \lambda(x_1', x_2', y_1, y_1)$ for all $(x_1, x_2, y_1, y_1), (x_1', x_2', y_1, y_1) \in A_0^R$.

Next, for any $(x_1, x_1, y_1, y_2) \in A_0^R$, there exists $x_2 \neq x_1$ such that $(x_1, x_2, y_1, y_2) \in A_0^R$. Indeed, as in the proof of Lemma 5, there is a neighborhood U of x_1 such that $\{x_2' \in X | (x_1, x_2', y_1, y_2) \in A_0\} \cap U$ is a differentiable manifold of dimension $n-1$. This set must contain an element x_2 such that $(x_1, x_2, y_1, y_2) \in A_0^R$, for otherwise $(\nabla_Y v)(x_2', y_1) - (\nabla_Y v)(x_1, y_1) = 0$ for all x_2' in the set, contradicting the second part of Condition 4B.

We can now conclude the proof of the lemma by extending the final step in the proof of Lemma 5: we show for arbitrary $(x_1, x_2, y_1, y_2), (x_1', x_2', y_1', y_2') \in A_0^R$ satisfying $x_1 \neq x_2$, $y_1 \neq y_2$, $x_1' \neq x_2'$ and $y_1' \neq y_2'$ that $\lambda(x_1, x_2, y_1, y_2) = \lambda(x_1', x_2', y_1', y_2')$. Assume w.l.o.g. $(\nabla_X v)(x_1, y_1) - (\nabla_X v)(x_1, y_2) \neq 0$, $(\nabla_Y v)(x_1, y_1) - (\nabla_Y v)(x_2, y_1) \neq 0$, $(\nabla_X v)(x_1', y_1') - (\nabla_X v)(x_1', y_2') \neq 0$ and $(\nabla_Y v)(x_1', y_1') - (\nabla_Y v)(x_2', y_1') \neq 0$. By Condition 4A, there is a sequence $(x_1^k)_k$ with $\lim_{k \to \infty} x_1^k = x_1$ satisfying $(x_1^k, x_1^k, y_1', y_2') \in A_0^R$ for all k. By continuity of $\nabla_X v$, there exists K such that $(x_1^k, x_2, y_1, y_2) \in A_0^R$ for all $k \geq K$. We get, for $k \geq K$:

$$\begin{aligned} &\lambda(x_1^k, x_2, y_1, y_2) = \lambda(x_1^k, x_1^k, y_1, y_2) = \lambda(x_1^k, x_1^k, y_1', y_2') \\ &= \lambda(x_1^k, x_2^k, y_1', y_2') = \lambda(x_1^k, x_2^k, y_1', y_1') = \lambda(x_1', x_2', y_1', y_1') = \lambda(x_1', x_2', y_1', y_2'), \end{aligned}$$

where $x_2^k \neq x_1^k$ is chosen such that $(\nabla_Y v)(x_1^k, y_1') - (\nabla_Y v)(x_2^k, y_1') \neq 0$, which is possible by the argument in the previous paragraph. Letting $k \to \infty$, the continuity of λ implies $\lambda(x_1, x_2, y_1, y_2) = \lambda(x_1', x_2', y_1', y_2')$. □

To conclude the proof of Theorem 1', note that for all $y_1 \neq y_2$ and $x_1 \neq x_2$, Lemma 7 and Condition 4A imply that there is a smooth path from x_2 to x_1 such that $(\nabla_X \gamma v)(x, y_1) - (\nabla_X \gamma v)(x, y_2) = \lambda_0((\nabla_X v)(x, y_1) - (\nabla_X v)(x, y_2))$ holds a.e. on the path. We hence obtain $F_{\gamma v}(x_1, x_2, y_1, y_2) = \lambda_0 F_v(x_1, x_2, y_1, y_2)$ from integration, as in the proof of i)⇒ ii) in Theorem 1. That h (g) is constant if $I < J$ ($I > J$) follows as in that proof, invoking Condition 4B instead of Condition 3. □

Proof of Theorem 2. Let $I \leq J$ (the proof for $I \geq J$ is analogous). Consider some $i \in \mathcal{I}$ and a given, fixed type profile for all other agents $(x_{-i}, y_1, ..., y_J)$. Given any such type profile, we re-order the workers and employers other than i such that $x^{(1)} \geq ... \geq x^{(I-1)}$ and $y^{(1)} \geq ... \geq y^{(J)}$.

We now verify the monotonicity condition identified by Bergemann and Välimäki.[25] This requires that the set of types of agent i for which a particular social alternative is efficient forms an interval. Let then m_k, $k = 1, ..., I$ denote the matching that matches $x^{(l)}$ to $y^{(l)}$ for $l = 1, ..., k-1$, x_i to $y^{(k)}$ and $x^{(l)}$ to $y^{(l+1)}$ for $l = k, ..., I-1$. Then, for $k = 2, ..., I-1$ it

[25] We only verify it for type profiles for which all these inequalities are strict. When some types coincide, it is still straightforward to verify monotonicity but we do not spell out the more cumbersome case distinctions here.

holds that the set

$$\{x_i \in X | u_{m_k}(x_1, ..., x_I, y_1, .., y_J) \geq u_m(x_1, ..., x_I, y_1, .., y_J) \ , \forall m \in \mathcal{M}\}$$

is simply $[x^{(k)}, x^{(k-1)}]$. For $k = I$ the set is $(\inf X, x^{(I-1)}]$, and for $k = 1$ it is $[x^{(1)}, \sup X)$. Monotonicity for workers j is verified in the same way.

Next, the necessary condition of Bergemann and Välimäki, spelled out for our matching model, requires that at all "switching points" $x_i = x^{(k-1)}$ where the efficient allocation changes, it also holds that

$$\frac{\partial}{\partial x_i}((\gamma v)(x_i, y^{(k-1)}) - (\gamma v)(x_i, y^{(k)})) \geq 0.$$

Given x_i and $y' > y$ we can always complete these to a full type profile such that x_i is a change point at which the efficient match for x_i switches from y to y'. Hence $\frac{\partial}{\partial x}((\gamma v)(x, y') - (\gamma v)(x, y)) \geq 0$ for all x and $y' > y$. So, γv must have increasing differences, i.e. it is supermodular. Since $\frac{\partial}{\partial x_i}((\gamma v)(x_i, y^{(k-1)}) - (\gamma v)(x_i, y^{(k)})) \geq 0$ is satisfied for all $x_i \in X$ (not just at switching points!), the second part of the sufficient conditions of Bergemann and Välimäki is satisfied. The argument for workers (yielding supermodularity of $(1 - \gamma)v$) is analogous. This completes the proof. □

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