Monotonic Norms and Orthogonal Issues in Multi-Dimensional Voting

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Abstract

We study issue-by-issue voting and robust mechanism design in multi-dimensional frameworks where privately informed agents have preferences induced by general norms. We uncover the deep connections between dominant strategy incentive compatibility (DIC) on the one hand, and several geometric/functional analytic concepts on the other. Our main results are: 1) Marginal medians are DIC if and only if they are calculated with respect to a basis such that the norm is *orthant-monotonic* in the associated coordinate system. 2) Equivalently, marginal medians are DIC if and only if they are computed with respect to coordinates determined by a *basis* such that, for any vector in the basis, any linear combination of the other vectors is *Birkhoff-James orthogonal* to it. 3) We show how *semi-inner products* and *normality* provide an analytic method that can be used to find all DIC marginal medians. 4) As an application, we derive all DIC marginal medians for l_p spaces of any finite dimension, and show that they do not depend on p (unless $p = 2$).

1 Introduction

We analyze a social choice/mechanism design problem where several privately informed agents take a multi-dimensional, collective decision. The main results identify the particular issues that can be put to vote in order to obtain robust mechanisms when issue-by-issue

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voting by (possibly qualified) majority is used to determined the outcome of the collective choice.

In our model, there are no monetary transfers, and the utility of each agent is determined by the distance between a privately known, individual peak (or ideal point) and the taken decision. This distance is derived from a norm on the decision space that is assumed here to be a vector space. The norm can vary across agents, may be private information, and it need not be generated by an inner product. In particular, it need not be the Euclidean norm.

Our main insights connect incentive compatible mechanisms and voting by (qualified) majority on the one side, to deep, elegant concepts and results from the geometry/functional analysis of *Minkowski* spaces (i.e., finite dimensional, real normed spaces) on the other.

With a few notable exceptions (see Literature review below), the literature on *incentive-compatible* multi-dimensional voting and its applications to Political Science and Economics has focused on models that use quadratic loss functions. When applied to normed vector spaces, this yields utilities derived from variations on the Euclidean norm (see for example the text book by Austen-Smith and Banks [2005])[1]. The reason behind this choice is technical : it allows the use of familiar mathematical methods from Euclidean geometry and/or familiar mean/variance statistical methods associated to the quadratic formulation.

But, quadratic loss functions derived from an Euclidean norm are not always suitable for multi-dimensional applications. Consider for example the choice of a budget on two items where each agent has a preferred ideal point. Then, under Euclidean distance, equal deviations on each item are perceived in the same way, equal deviations upwards and downwards from the wished spending on one item are also perceived in the same way, and, moreover, since utility is separable in the two dimensions, there is no cross-interaction among spending deviations on the two items . While it is possible to extend some of the results based on Euclidean norm to the more general class of quadratic preferences generated by inner-products, and to hereby address some of the above raised issues, there is an obvious need to understand more general models where preferences do not display such a high degree of symmetry[2]. Moreover, as we shall prove below, inner-product norms constitute, technically, a very special and a-typical case, that cannot fully capture the basic structure of the underlying problem. For example, inner-product norms do not reveal the fundamental difference between the geometries of two-dimensional and higher-dimensional spaces, and its implications for mechanism design.

Even for purely location problems (of a facility, say) the relevant distance function need not be Euclidean since transport costs must take into account, for example, traveling time that may be a function of local topography, existing or planned infrastructure, etc..Think

[1] The same holds for many other related literatures (e.g., on signaling and cheap talk), and for empirical methodologies (see for example, Clinton et al [2004])

[2] See Eguia[2013] for a critical discussion of these issues and for references to papers that empirically test this and alternative distance functions.

about the proverbial cab driver in Manhattan who needs to use the "taxicab" norm, driving along the right angles imposed by the array-like city street map. Indeed, distance in US cities is often colloquially measured in "blocks". Such a distance function, for example, is not generated by an inner-product norm.

We focus here on issue-by-issue voting by majority: this yields as outcome the issue-by-issue (or marginal, or coordinate-wise) median. The marginal median is the prime example for a "structure induced" equilibrium in the spirit of Shepsle [1979].[3] Besides its ubiquity in practice, this type of voting mechanism (together with its generalization to the so-called "generalized medians" that allow for the presence of additional "phantom" voters with fixed, known peaks) exhaust the set of dominant-strategy incentive compatible (DIC henceforth) mechanisms in various settings where the preference domain is sufficiently rich. The first fundamental result in this vein was obtained by Moulin [1980] in the one-dimensional case.[4] Common examples of generalized medians are obtained by issue-by-issue voting with a qualified majority that may differ from dimension to dimension (e.g., a decision where one aspect requires a constitutional amendment and hence a higher majority). Our analysis easily generalizes to such mechanisms as well.

Due to the multi-dimensionality of the decision space, the issues on which voting can take place are not uniquely defined. In other words, the issues that are put on the ballot are endogenous. Technically, every algebraic basis for the space of decision vectors defines a set of issues (or coordinates) along which issue-by-issue voting by majority can be conducted to yield a combined, feasible decision. Since the median is not a linear function, the obtained multi-dimensional marginal (or coordinate-wise) median varies with the underlying system of coordinates (see Haldane [1948]). Here we want to find those particular systems of coordinates that induce agents to report truthfully if a marginal median with respect to those coordinates is used.

Our first main result shows that, with preferences induced by norms, marginal medians are DIC if and only if they are calculated with respect to a basis such that the norm is *orthant-monotonic* in the associated coordinate system. Orthant-monotonicity is a strictly weaker version of the *monotonicity* condition for norms. Monotonicity compares all possible pairs of vectors that are ordered with respect to the lattice structure of the underlying space and requires that the norm of a vector with larger coordinates (in absolute values) is larger[5]. Orthant-monotonicity applies the same condition, but requires it to hold only for pairs of ordered vectors in the same orthant.

Barbera, Gul and Stacchetti [1993] (BGS henceforth) assumed that the underlying out-

[3] Shepsle proposed these as a response to the lack of equilibria in multi-dimensional models where voting is not formally constrained by institutional arrangements. See also Feld and Grofman [1988] and Kramer [1972],[1973].

[4] See Gershkov, Moldovanu and Shi [2017] and Kleiner and Moldovanu [2017] for implementation of generalized medians by means of sequentially binary procedures with varying majority requirements. Phantoms are then not required.

[5] This yields a *Riesz space* equipped with a *lattice norm*

come set is a product of lines, and they fixed a system of directions, but did not focus on norm-based preferences. Instead, they studied a richer class of preferences called *multidimensional singled peaked (m.s.p).* BGS showed that, on the class of m.s.p. preferences a mechanism is DIC if and only if it is a generalized marginal median[6]. BGS also showed that their class is maximal in the sense that, if an agent has a preference outside it, there exists a marginal median that is not DIC. In an earlier paper, Border and Jordan [1983] considered a different rich domain of preferences which they called *star-shaped and separable*, and obtained similar results[7].

We show that a norm-based preference is m.s.p. in the BGS sense if and only if the underlying norm is orthant monotonic, and that it is star-shaped and separable in the Border-Jordan sense if and only if it is monotonic. While our analysis focuses on the dependence on the chosen coordinate system – since norm monotonicity properties crucially depend on the underlying coordinate system –, this dependence does not play a role neither in the BGS' nor in Border and Jordan's analysis. We shall explain below this discrepancy in terms of the different domains and focuses of the respective studies.

Given the above characterizations, we need to understand how to actually find the coordinate systems (if any) that yield an orthant-monotonic representation of a given, fixed norm. For the Euclidean norm, Kim and Roush [1984] and Peters, van der Stel, and Storcken [1992] connected the DIC property of marginal medians to *orthogonal* coordinate systems. We do indeed show that orthant-monotonicity of the standard Euclidean norm is equivalent to requiring that the underlying coordinate system is defined by an orthogonal basis.

Another elegant result due to Peters, van der Stel, and Storcken [1993] shows that marginal medians constitute DIC mechanisms for a general norm on the plane (i.e., when there are two dimensions) if and only if majority voting takes place along two directions that are *mutually orthogonal* in the sense of Birkhoff and James (Birkhoff [1935], James [1947], BJ henceforth)[8]. This last result offers an important additional hint for our search, but, as we show below, it holds in this form only for two-dimensional normed spaces.

The BJ-orthogonality relation can be applied to any normed vector space, and it reduces to the usual orthogonality relation in Hilbert spaces, i.e., two vectors are then orthogonal when their *inner-product* equals zero. Roughly speaking, BJ-orthogonality applies to convex sets that are point-symmetric around a center (and hence can serve as unit balls of a norm) the insight that a radius is orthogonal to the tangent through the point where the radius hits the circle's boundary[9].

[6] Assuming a rich set of preferences, Nehring and Puppe [2007]studed generalized medians on very general abstract domains called "median spaces". These do not necessarily have a vector space structure.

[7] Zhou [1991], and Barbera and Jackson [1994] characterize DIC mechanisms on the larger class of continuous, strictly quasi-concave utility functions with a unique maximizer. The range of such a mechanism must be one-dimensional.

[8] Stel [2000] extends these insights to non-anonymous mechanisms.

[9] See Alonso [1988] and Martini[2001] for excellent surveys of this and other related topics.

The main "defects" of BJ-orthogonality, are a lack of both symmetry and additivity[10] – as we shall see below, both these defects have important consequences for our mechanism design analysis, and need to be "corrected" here.

The existence of BJ-mutually orthogonal vectors involves a fixed-point argument, and is therefore not obvious unless the space is Hilbert, where the orthogonality relation is symmetric. For two-dimensional spaces equipped with a strictly convex norm, Peters, van der Stel, and Storcken [1993] contructed a BJ-mutually orthogonal pair of vectors, and therefore proved the existence of at least one DIC marginal median mechanism.

For any two-dimensional normed space we prove here the existence of at least **two** distinct DIC marginal medians. This is done by invoking a famous result that goes back to Hermann Auerbach[11]: he showed that any convex body that is point-symmetric around a center possess at least two pairs of *conjugate diameters.* The directions of the diameters in each pair define BJ-mutually orthogonal pairs of vectors.

In more than two dimensions, an algebraic basis consisting of BJ-mutually orthogonal vectors such that each vector in the basis is orthogonal to any linear combination of the others is called an *Auerbach basis.* Auerbach's construction does generalize to any number of dimensions, and the two constructed bases always possess this *additivity on the right* property[12][13].

Interestingly enough, with more than two dimensions, an Auerbach basis as defined above is not sufficient in order to induce a DIC marginal median! Our next main result shows that orthant-monotonicity (and hence the DIC property of marginal medians) is equivalent to a *additivity on the left* property: marginal medians are DIC if and only if they are computed with respect to a basis such that, for any vector in the basis, any linear combination of the other vectors is also BJ-orthogonal to it.

To understand the intuition behind this result, consider a three-dimensional normed space, and choose a set of three mutually orthogonal issues (this always exists by Auerbach's construction). Assume that a decision has already been taken on two issues (by consecutive majority votes, say): the obtained decision can be any arbitrary vector in the respective two-dimensional subspace. DIC requires that the remaining third issue be BJ orthogonal to the already taken decision. This property is automatically satisfied in Hilbert space if we started the whole process with an orthogonal basis for the entire space, but is not satisfied in general, and it needs to be additionally imposed.

Mechanism design exercises usually call for the maximization of a given goal over a

[10] If BJ orthogonality is symmetric or additive and the dimension is at least three, the space must be Hilbert (see James [1947] and [1987]).

[11] The result is attributed to H. Auerbach .by Stephan Banach. But Auerbach was killed by the Nazis, and his dissertation was burnt. Hence there is no trace of the original proof.

[12] Additivity on the right of the BJ-orthogonality relation is automatically satisfied if the norm is smooth. This is a consequence of the semi inner-product representation and normality - see below.

[13] Peters et al's two-dimensional construction yields indeed one of the Auerbach bases. They generally invoke strict convexity of the norm, but this is not necessary at this step.

sufficiently broad class of mechanisms that satisfy certain properties. For such purposes it is of interest to first identify **all** pairs of BJ mutually orthogonal vectors. While this is a relatively straightforward task in Hilbert spaces, the situation radically changes for general normed spaces, as we describe below.

Gershkov, Moldovanu and Shi [2018] maximize utilitarian welfare over the class of marginal median mechanisms in Hilbert spaces.[14] Technically, they maximize over the continuum-size, multiplicative group of linear *isometries*, i.e., maps that preserve distance, such as rotations. In inner-product spaces a linear isometry also preserves angles, and hence orthogonality.[15] Thus, applying an isometry to a given, arbitrary pair of orthogonal issues yields another such pair: because medians are not linear functions, and hence not necessarily isometry-covariant, this operation may yield a distinct marginal median.[16] Moreover, applying all isometries to any given orthogonal pair exhausts the set of all relevant orthogonal pairs, and hence the set of all DIC marginal medians (see Kim and Roush [1984] and Peters, van der Stel, and Storcken [1992] for characterizations of DIC mechanisms in terms of orthogonality and isometries for the Euclidean norm).

Although BJ-orthogonality preserving maps do reduce to isometries in general normed spaces (this is a relatively recent result due to Koldobsky [1993]), it is well known that any finitely dimensional normed vector space that is **not** Hilbert admits only a **finite** number of isometries. Moreover, it is not the case that applying all possible isometries to a given pair of BJ-mutually orthogonal directions yields all other possible such pairs. For example, there is no isometry that maps one Auerbach basis into another where the two bases stem from the Auerbach construction mentioned above.[17] Hence, a purely geometric approach cannot yield all mutually orthogonal vectors, and thus is not too helpful in exercises where a comprehensive class of incentive-compatible mechanisms needs to be first identified.

To overcome the above difficulty, we employ the *semi-inner products* (SIP), due to Lumer [1961], and further developed by Giles [1967]. An SIP is a bivariate form that can be defined for any pair of vectors in any normed space: it resembles an inner product, but is neither symmetric, nor additive on the right. Giles has shown that, for the class of smooth (i.e., *Gateaux-differentiable*) norms, BJ-orthogonality coincides with *normality* which means that the SIP equals zero. Moreover, a *norm-consistent SIP* is then unique and, importantly for our purposes, has an **analytic** formulation in terms of the underlying norm functional and its directional derivatives. This analytic approach can, in principle, be used to obtain all pairs of mutually orthogonal vectors – and hence, in principle, all DIC marginal median mechanisms – as the set of solutions to a system of non-linear equations.

As a main application of the SIP approach, we characterize, for any finite dimension $d \geq 2$ and for any $p \geq 1$, the bases (or issues) yielding DIC marginal medians for any

[14] For two dimesnions, this coincides with the class of anonymous, Pareto-optimal and DIC mechanisms. With more dimensions, marginal medians need not be Pareto optimal.

[15] The famous Mazur-Ulam [1932] theorem asserts that any surjective isometry must be linear.

[16] Medians are translation covariant, so it is eough to consider isometries that fix the origin.

[17] In this sense, Euclid's *fourth postulate* does not hold in normed spaces that are not Hilbert. T

$l_p(d)$ space. We also show that these do not depend at all on p (unless $p = 2$, the only Hilbert space in this class).[18] Thus, the characterized marginal medians remain DIC even for situations where the norm is allowed to vary across agents within the l_p class, and is their private information-

For $d = 2$ and $p > 1$ our results imply that are exactly **two** distinct DIC marginal mechanisms for a $l_p(d)$ space, and hence (by the result of Peters, van der Stel, and Storcken [1992]) exactly **two** DIC, anonymous and Pareto optimal mechanisms: these correspond to 0 and 45 degrees rotations of the Cartesian axes. It is interesting to note that the welfare analysis of Gershkov, Moldovanu and Shi [2018] for the Euclidean $l_2(2)$ case focused precisely on these rotations.

Analogous robust results can be obtained for other classes of norms by identifying – via the SIP approach – the set of left-additive bases formed by mutually orthogonal vectors that are shared by all norms in the class. To further illustrate this approach, we consider a setting where agents use individually *weighted* Euclidean norm that is their private information. All these norms are generated by inner-products, and do not allow for cross-interactions among issues. Then, under a genericity condition on the set of possible weights, there is exactly **one** DIC marginal median mechanism, i.e., the Cartesian coordinates are the unique jointly orthogonal ones for all the norms in this class. Introducing even the slightest degree of interaction among the issues – by allowing utility functions derived from other, more general, inner-product norms – yields an impossibility result. These results are related to those of Border and Jordan [1983] (who did not consider norms or orthogonality).

The remainder of the paper is organized as follows: Section 2 presents the social choice model and marginal medians mechanisms. Section 3 introduces monotonicity properties of norms and connects DIC to orthant monotonicity. We also relate our insights to those obtained by BGS and Border and Jordan. Section 4 connects orthant-monotonicity (and hence DIC mechanisms) to bases consisting of BJ-mutually orthogonal vectors that satisfy a left-additivity condition. Section 5 shows how to analytically use semi-inner products in order to find all DIC marginal medians. Section 6 illustrates the various concepts and findings for l_p norms and for inner-product spaces, including several robust mechanism design results. Section 7 concludes.

2 The Social Choice Model

We consider an odd number of agents n who collectively choose a decision $\mathbf{v} \in V$, where V is a d-dimensional Minkowski (i.e., over the reals) vector space. Since any d-dimensional normed space over the reals is isomorphic to the space $\mathbb{R}^d$, we shall assume here w.l.o.g. that $V = \mathbb{R}^d$, and endow this space with different norms[19].

[18] The case $p < 1$ does not yield a normed space, and it is not considered here.

[19] Note that the isomorphism does depend on the assumed basis - we shall make this explicit below.

Throughout of the paper, the **bold** font is used to denote vectors in $\mathbb{R}^d$. We use $i = 1, ..., n$ to label voters, and j or $k = 1, ..., d$ to label coordinates.

We denote by $\{\mathbf{x}^1, ..., \mathbf{x}^d\}$ a generic algebraic basis for $\mathbb{R}^d$, where $\mathbf{x}^1, ..., \mathbf{x}^d$ are linearly independent, and by $\{\mathbf{e}^1, ..., \mathbf{e}^d\}$ the standard Cartesian basis where only the $j-th$ coordinate is different from zero, and equals one:

$$\mathbf{e}^j = (\underbrace{0, 0, .., 1}_{j}, 0, ..0)$$

Each agent's ideal position is given by a "peak" $\mathbf{t}_i \in \mathbb{R}^d$, $i = 1, 2, ..., n$. The peak $\mathbf{t}_i$ is agent i's private information. The utility of agent i with peak $\mathbf{t}_i$ from decision $\mathbf{v}$ is given by

$$-\left(\|\mathbf{t}_i - \mathbf{v}\|\right)$$

where $\|\cdot\|$ is a *norm* on $\mathbb{R}^d$. Recall that a norm $\|\cdot\|$ is a real-valued, non-negative function on $\mathbb{R}^d$ that satisfies:

1. $\|\mathbf{x}\| \geq 0$;
2. $\|\mathbf{x}\| = 0 \Leftrightarrow \mathbf{x} = \mathbf{0}$;
3. $\|a\mathbf{x}\| = |a| \|\mathbf{x}\|, \forall \mathbf{x} \in \mathbb{R}^d, \; a \in \mathbb{R}$;
4. $\|\mathbf{x} + \mathbf{y}\| \leq \|\mathbf{x}\| + \|\mathbf{y}\|, \forall \mathbf{x}, \mathbf{y} \in \mathbb{R}^d$.

Since our analysis and results are purely ordinal, they immediately apply to all utility functions of the form

$$-\Delta\left(\|\mathbf{t}_i - \mathbf{v}\|\right)$$

where Δ is a strictly monotonic function: all these cardinal utilities represent the same ordinal preferences as the basic norm $\|\cdot\|$.

We do **not** assume that the norm inducing the above utility functions is generated by an inner-product, i.e., the vector space need not be a Hilbert space.

2.1 Marginal Medians

For the following properties we assume that mechanisms only depend on reported peaks, which holds by definition for the (generalized) marginal medians we consider.

Definition 1 *1. A direct revelation mechanism is a function* $\psi : (\mathbb{R}^d)^n \to \mathbb{R}^d$.

2. *A direct revelation mechanism* $\psi(\mathbf{t}_i, \mathbf{t}_{-i})$ *is dominant-strategy incentive compatible (DIC) if for any voter i, for any realizations* $\mathbf{t}_i$ *and* $\mathbf{t}_{-i}$ *and for any reporting strategy profile* $\hat{\mathbf{t}}_{-i}(\mathbf{t}_{-i})$ *of the other voters it holds that*

$$\left\|\mathbf{t}_i - \psi\left(\mathbf{t}_i, \hat{\mathbf{t}}_{-i}(\mathbf{t}_{-i})\right)\right\| \leq \left\|\mathbf{t}_i - \psi\left(\hat{\mathbf{t}}_i, \hat{\mathbf{t}}_{-i}(\mathbf{t}_{-i})\right)\right\|, \; \forall \hat{\mathbf{t}}_i$$

3. A direct revelation mechanism is anonymous if $\psi(\mathbf{t}_1, .., \mathbf{t}_n) = \psi(\sigma(\mathbf{t}_1, .., \mathbf{t}_n))$ *for all* $\mathbf{t}_1, ..., \mathbf{t}_n \in \mathbb{R}^d$ *, and for all permutations* σ*..*

In words, DIC says that any individual manipulation forces the social choice to move away from the true peak, as measured by the distance function used by the manipulating agent. Note that the DIC constraint for agent i only uses the norm considered by agent i, and hence we can easily generalize the above definition to situations where agents use different norms.

Let $\{\mathbf{x}^1, ..., \mathbf{x}^d\}$ be an algebraic basis for $\mathbb{R}^d$. Then each $\mathbf{y} \in \mathbb{R}^d$ can be represented as

$$\mathbf{y} = \sum_{j=1}^{d} \alpha^j(\mathbf{y})\mathbf{x}^j,$$

where $\alpha^j(\mathbf{y})$, $j = 1, ..., d$, is the j-th coordinate of $\mathbf{y}$ with respect to this basis.

Definition 2 *The marginal median mechanism (MMM) with respect to basis* $\{\mathbf{x}^1, ..., \mathbf{x}^d\}$ *is defined as*

$$\psi(\mathbf{t}_1, .., \mathbf{t}_n) = \sum_{j=1}^{d} med(\alpha^j(\mathbf{t}_1), ..., \alpha^j(\mathbf{t}_n))\mathbf{x}^j$$

where $med(\alpha^j(\mathbf{t}_1), ..., \alpha^j(\mathbf{t}_n))$ *is the median of the* j*-th coordinates of the agents' peaks.*

All our results below can be extended to *generalized medians* that are obtained by setting a fixed number of "phantoms" peaks at some commonly known locations, and then taking the marginal median among the reported peaks of the real agents and the commonly known phantom peaks. (Generalized) marginal medians are anonymous, but need not be Pareto optimal[20].

3 Incentive Compatibility and Monotonic Norms

In this section, we first define norm monotonicity and orthant monotonicity with respect to a given algebraic basis in $\mathbb{R}^d$. We then show that a marginal median mechanism computed with respect to the coordinate system associated with a given basis is DIC if and only if the norm is orthant monotonic with respect to that same basis. Finally, we discuss how this result is related to the insights in Barbera, Gul and Stacchetti [1993] and Border and Jordan [1983].

[20] The standard marginal median (corresponding to voting by simple majority in each dimension) is Pareto optimal for $d = 2$

3.1 Orthant Monotonicity and DIC Marginal Medians

Fix an algebraic basis in $\mathbb{R}^d$ consisting of d linearly independent vectors $\{\mathbf{x}^1, ..., \mathbf{x}^d\}$. Recall that we can represent each $\mathbf{x} \in \mathbb{R}^d$ as

$$\mathbf{x} = \sum_{j=1}^{d} \alpha^j(\mathbf{x})\mathbf{x}^j,$$

where $\alpha^j(\mathbf{x})$ is the j-th coordinate of $\mathbf{x}$ according to this basis. To simplify notation when confusion cannot arise, we write

$$(x_1, x_2, ..., x_d) = \left(\alpha^1(\mathbf{x}), \alpha^2(\mathbf{x}), ..., \alpha^d(\mathbf{x})\right)$$

and identify $\mathbf{x}$ with the vector of coordinates $(x_1, x_2, ..., x_d)$.

It is important to note that the monotonicity properties described below depend on the underlying coordinate system.

Definition 3 *A norm $\|\cdot\|$ on $\mathbb{R}^d$ is monotonic (Bauer, Stoer and Witzgall [1961]) if*

$$\|(x_1, x_2, ..., x_d)\| \leq \|(y_1, y_2, ..., y_d)\|$$

whenever

$$|x_j| \leq |y_j| \text{ for all } j = 1, ..., d.$$

A norm $\|\cdot\|$ on $\mathbb{R}^d$ is orthant-monotonic (Gries [1967]) if

$$\|(x_1, x_2, ..., x_d)\| \leq \|(y_1, y_2, ..., y_d)\|$$

whenever

$$x_j y_j \geq 0 \text{ and } |x_j| \leq |y_j| \text{ for all } j = 1, ..., d.$$

It is clear from the above definition that monotonicity implies orthant-monotonicity.[21] The following lemma provides some useful characterizations that we repeatedly use below (see Johnson and Nylen [1991] or Horn and Johnson [2013], p.340 for these and other monotonicity properties of norms):

Lemma 1 *1. A norm is monotonic if and only if it is absolute:*

$$\|(x_1, x_2, ..., x_d)\| = \|(|x_1|, |x_2|, ..., |x_d|)\|$$

for all $\mathbf{x} \in \mathbb{R}^d$.

2. A norm is orthant-monotonic if and only if it satisfies

$$\|(x_1, ..., x_{j-1}, 0, ..., x_d)\| \leq \|(x_1, ..., x_{j-1}, x_j, ..., x_d)\|$$

for all $\mathbf{x} \in \mathbb{R}^d$ and all j.

[21] A monotonic norm can also be seen as a *lattice norm* since it is consistent with the standard partial order on $\mathbb{R}^d$. Thus, a normed space endowed with a monotonic norm becomes a *Riesz space.*

Example 1 *Fix an algebraic basis* $\{\mathbf{x}^1, \mathbf{x}^2, ..., \mathbf{x}^d\}$*. For* $\mathbf{x} = (x_1, ..., x_d)$ *and* $p \geq 1$*, define*

$$\|\mathbf{x}\|_p = \left(\sum_{j=1}^{d} |x_j|^p \right)^{1/p}.$$

This is the class of l_p *norms with respect to the given basis. All these norms are absolute, hence monotonic and hence orthant-monotonic. The same holds for the limit norm*

$$\|\mathbf{x}\|_\infty = \max_j |x_j|.$$

Example 2 *Let* $d = 2$*, and fix the standard Cartesian basis.*

1. *Consider the norm with unit ball defined as the hexagon with vertices at* $\pm(1,1), \pm(1,0)$ *and* $\pm(0,1)$*. This norm is orthant-monotonic but* ***not*** *monotonic. For example,*

$$\|(1,-1)\| > 1 = \|(1,1)\|$$

which contradicts the fact that a monotonic norm can only depend on absolute values (Lemma 1-1).

2. *Consider the norm with unit ball defined as the parallelogram with vertices at* $\pm(2,2)$ *and* $\pm(1,-1)$*. This norm is* ***not*** *orthant-monotonic. For example,*

$$\|(2,0)\| > 1 = \|(2,2)\| \tag{1}$$

which contradicts the characterization in Lemma 1-2.

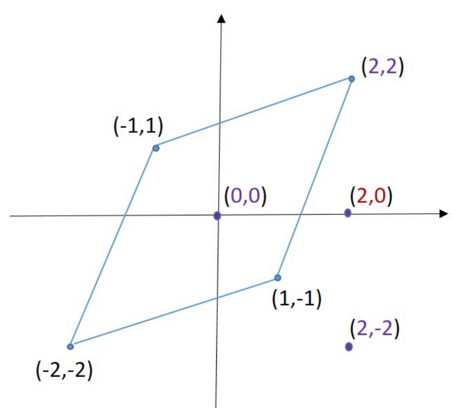

Figure 1: A non-orthant-monotonic norm

To see the consequences of non-orthant monotonicity on marginal medians, consider three agents with peaks at $(0,0)$*,* $(2,2)$ *and* $(2,-2)$*, respectively. If all agents report truthfully, the marginal median (with respect to the fixed Cartesian coordinates) is* $(2,0)$*. If the agent with peak at* $(0,0)$ *deviates and reports instead* $(2,2)$*, the marginal median becomes* $(2,2)$*. By inequality (1), this deviation is profitable for this agent. In contrast, marginal mechanisms are DIC if computed with respect to the coordinates defined by the basis* $\{(-1,1),(1,1)\}$ *or by the basis* $\{(1,3),(1,1/3)\}$*.*[22]*.*

[22]We show below that there are always at least two bases with this property.

We can now state our first main result:

Theorem 1 *A marginal median mechanism is DIC if and only if it is computed with respect to a basis* $\{\mathbf{x}^1, \mathbf{x}^2, ..., \mathbf{x}^d\}$ *such that the norm is orthant-monotonic in the associated coordinate system.*

Proof. (If direction). Fix $\{\mathbf{x}^1, ..., \mathbf{x}^d\}$ to be a basis with the required property. We show that the MMM with respect to this basis is DIC. For any vector $\mathbf{x}$,let $\sum_{j=1}^{d} \alpha^j(\mathbf{x})\mathbf{x}^j$ be its representation in the fixed basis. Let $\mathbf{t}_1$ be agent $1's$ true peak, and consider a deviation to $\widetilde{\mathbf{t}}_1$. Assume that the other agents ($i = 2, ..., n$) make arbitrary reports $\mathbf{t}_2, ..., \mathbf{t}_n$.

Let $\mathbf{m} = \mathbf{m}(\mathbf{t}_1, ..., \mathbf{t}_n)$ and $\widetilde{\mathbf{m}} = \mathbf{m}(\widetilde{\mathbf{t}}_1, ..., , \mathbf{t}_n)$ be the marginal medians when agent 1 reports truthfully and when he deviates to $\widetilde{\mathbf{t}}_1$, respectively. The argument for any other agent is analogous. We need to show that

$$\|\mathbf{t}_1 - \mathbf{m}\| \leq \|\mathbf{t}_1 - \widetilde{\mathbf{m}}\|,$$

or equivalently,

$$\left\| \sum_{j=1}^{d} (\alpha^j(\mathbf{t}_1) - \alpha^j(\mathbf{m}))\mathbf{x}^j \right\| \leq \left\| \sum_{j=1}^{d} (\alpha^j(\mathbf{t}_1) - \alpha^j(\widetilde{\mathbf{m}}))\mathbf{x}^j \right\|. \tag{2}$$

By the properties of the one-dimensional median we obtain that, for any $j = 1, 2, ..., d$,

$$\text{either } \ \alpha^j(\mathbf{t}_1) \leq \alpha^j(\mathbf{m}) \leq \alpha^j(\widetilde{\mathbf{m}}) \ \text{ or } \ \alpha^j(\widetilde{\mathbf{m}}) \leq \alpha^j(\mathbf{m}) \leq \alpha^j(\mathbf{t}_1). \tag{3}$$

This immediately implies that, for all $j = 1, 2, ..., d$,

$$\left(\alpha^j(\mathbf{t}_1) - \alpha^j(\mathbf{m})\right)\left(\alpha^j(\mathbf{t}_1) - \alpha^j(\widetilde{\mathbf{m}})\right) \geq 0, \tag{4}$$

and that

$$\left|\alpha^j(\mathbf{t}_1) - \alpha^j(\mathbf{m})\right| \leq \left|\alpha^j(\mathbf{t}_1) - \alpha^j(\widetilde{\mathbf{m}})\right|. \tag{5}$$

Therefore, (2) follows immediately by applying orthant monotonicity to the two vectors

$$\sum_{j=1}^{d} (\alpha^j(\mathbf{t}_1) - \alpha^j(\mathbf{m}))\mathbf{x}^j \text{ and } \sum_{j=1}^{d} (\alpha^j(\mathbf{t}_1) - \alpha^j(\widetilde{\mathbf{m}}))\mathbf{x}^j.$$

(Only if direction). The proof is a generalization of the insight in Example 2. Assume that the MMM is computed with respect to a basis $\{\mathbf{x}^1, ..., \mathbf{x}^d\}$ whose coordinate system does not yield an orthant-monotonic representation of the norm. By Lemma 1 there exists k, a vector $\mathbf{x} = \sum_{j \neq k} \alpha^j(\mathbf{x})\mathbf{x}^j \neq 0$ and a scalar β such that

$$\|\mathbf{x}\| = \left\|\alpha^1(\mathbf{x}), ..., \alpha^{k-1}(\mathbf{x}), 0, ..., \alpha^d(\mathbf{x})\right\| > \left\|\alpha^1(\mathbf{x}), ..., \alpha^{k-1}(\mathbf{x}), \beta, ..., \alpha^d(\mathbf{x})\right\| = \left\|\mathbf{x} + \beta\mathbf{x}^k\right\|$$

Consider now the following profile of preferences:

$$\begin{aligned}
\mathbf{t}_i &= \mathbf{x}-\beta\mathbf{x}^k,\ i=1,...,\frac{n-1}{2} \\
\mathbf{t}_i &= \mathbf{x}+\beta\mathbf{x}^k,\ i=\frac{n+1}{2},...,n-1 \\
\mathbf{t}_i &= \mathbf{0},\ i=n
\end{aligned}$$

Hence, if every agent reports truthfully, we obtain that

$$\mathbf{m}(\mathbf{t}_1,...,\mathbf{t}_n)=\mathbf{x}.$$

Consider now a deviation of agent n to $\widetilde{\mathbf{t}}_n=\mathbf{x}+\beta\mathbf{x}^k$ which yields

$$\mathbf{m}(\mathbf{t}_1,...,\widetilde{\mathbf{t}}_n)=\mathbf{x}+\beta\mathbf{x}^k.$$

Since by assumption $\|\mathbf{x}\|>\left\|\mathbf{x}+\beta\mathbf{x}^k\right\|$, this deviation is profitable for agent n who has a true peak at the origin. This is a contradiction to the assumption that the marginal median with respect to this basis is DIC for agent n. ■

The above proof shows that the sufficiency of orthant monotonicity for incentive compatibility is intimately linked to the main properties of the of the one-dimensional median. In one dimension, a deviation has either no effect on the median, or moves it farther away from the agent's peak: the old median under truth telling lies between the agent's peak and the new median after the deviation (formally (3)). By applying this observation, dimension by dimension, to a d-dimensional marginal median, we can conclude that the two difference vectors, one between the ideal point and the old d-dimensional median and the other between the ideal point and the new median, lie in the same orthant (formally (4)). Moreover, the later has larger corodinates than the former (formally (5)). Therefore, by orthant monotonicity, the agent's ideal point is nearer to the old median than to the new median.

Since the one-dimensional generalized medians (with possible phantoms) share the above property with the standard one-dimensional median, the above results easily extends to generalized marginal median mechanisms.

Remark 1 *Note that in the proof of sufficiency (if direction), we can replace the norm* $\|\cdot\|$ *by individual-specific norms* $\{\|\cdot\|_i\}$*, and the proof still goes through. This implies that an MMM remains DIC when agents' preferences are generated by possibly different norms that are all orthant monotonic with respect to a given basis. We use this observation below when we construct robust mechanisms for situations where the preference-inducing norms differ and are private information.*

3.2 Relations to Barbera, Gul and Stacchetti [1993] and to Border and Jordan [1983]

Barbera, Gul and Stacchetti [1993] studied a class of preferences called *multidimensional singled peaked (m.s.p.)*. BGS showed that, on that class of m.s.p. preferences (that are not necessarily induced by a norm!), a mechanism is DIC if and only if it is a generalized marginal median. They also showed that the m.s.p. class is maximal in the sense that, if an agent has a preference outside it, there exists a marginal median which is not DIC.

Border and Jordan [1983] characterized DIC mechanisms on a different domain of preferences that they called *star-shaped and separable.* A norm-based preference is always (weakly) star-shaped in the sense of Border and Jordan: this follows by the convexity of the norm functional (i.e., by the triangle inequality).[23] Here are their definitions (where we slightly re-formulate the BGS one to best highlight the connection with our normed spaces):

Definition 4 *1. A preference relation with ideal point* $\mathbf{x}$ *is m.s.p. if for every* $\mathbf{y}$ *and for every* $\mathbf{z}$ *on a shortest* l_1 *path from* $\mathbf{x}$ *to* $\mathbf{y}$, $\mathbf{z}$ *is (weakly) preferred to* $\mathbf{y}$.[24]

2. A preference induced by a norm $\|\cdot\|$ *is separable if, for all* $j = 1, ..., d$ *and for all* $x_{-j}, y_{-j}, x_j,$ *and* x'_j,

$$\|(x_j, x_{-j})\| \geq \left\|(x'_j, x_{-j})\right\| \Leftrightarrow \|(x_j, y_{-j})\| \geq \left\|(x'_j, y_{-j})\right\|.$$

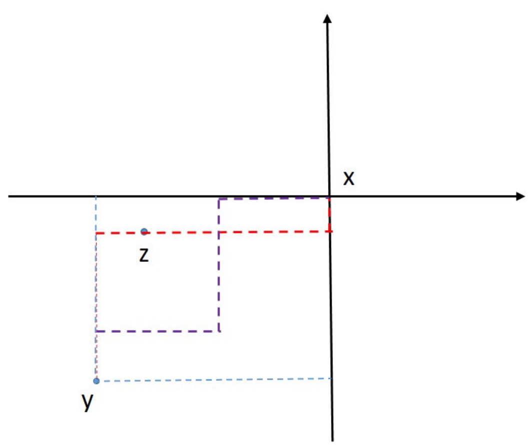

Figure 2: Point $\mathbf{z}$ is on a shortest l_1 path from $\mathbf{x}$ to $\mathbf{y}$

Given the BGS maximality result, and given our Theorem 1 above, we should be able to relate monotonicity properties of norms to the above general definitions.

Proposition 1 *Fix a coordinate system determined by a basis* $\{\mathbf{x}^1, ..., \mathbf{x}^d\}$, *and define the standard* l_1 *norm with respect to these coordinates:*

$$\|\mathbf{x}\|_1 = \sum_{j=1}^{d} \left|x^j\right| \quad \textit{for any } \mathbf{x} = \sum_{j=1}^{d} x_j \mathbf{x}^j.$$

[23] Strictly convex norms - see the definition in Section 6 - are (strongly) star-shaped.

[24] This definition implicitly assumes a fixed, given basis for calculationg the l_1 norm, e.g., the standard Euclidean one.

1. *The preference relation induced by a norm $\|\cdot\|$ is m.s.p. if and only if the norm is orthant-monotonic with respect to the chosen system of coordinates.*

2. *The preference relation induced by a norm $\|\cdot\|$ is separable if and only if the norm is monotonic with respect to the chosen system of coordinates.*

Proof. 1. Since the $\|\cdot\|$-based preference of an agent with peak at $\mathbf{x}$ is a translation of the preference of an agent with peak at $\mathbf{0}$, we can w.l.o.g. assume below that peaks are at $\mathbf{0}$.

Assume first that $\|\cdot\|$ is orthant-monotonic with respect to the fixed coordinates. Consider any $\mathbf{y}$ and any $\mathbf{z}$ on a shortest l_1 path from $\mathbf{0}$ to $\mathbf{y}$. Then $\mathbf{y}$ and $\mathbf{z}$ must be in the same orthant (see also picture above) and $|z_j| \leq |y_j|$ for all $j = 1, ..., d$.[25] Hence, $\|\mathbf{z}\| \leq \|\mathbf{y}\|$ by orthant-monotonicity, and $\mathbf{z}$ is preferred to $\mathbf{y}$ by an agent with peak at $\mathbf{0}$, yielding m.s.p.

Conversely, assume that the $\|\cdot\|$-based preference is m.s.p. Consider $\mathbf{y}$ and $\mathbf{z}$ such that $z_j y_j \geq 0$ and $|z_j| \leq |y_j|$ for all $j = 1, ..., d$. Then it is easy to see that $\mathbf{z}$ must lie on a shortest l_1 path from $\mathbf{0}$ to $\mathbf{y}$, and hence it must be preferred to $\mathbf{y}$. Since the preference is derived from the norm $\|\cdot\|$, we must have $\|\mathbf{z}\| \leq \|\mathbf{y}\|$ and thus the norm is orthant-monotonic.

2. Assume first that the norm $\|\cdot\|$ is monotonic, and assume that $\|(x_j, x_{-j})\| \geq \left\|(x'_j, x_{-j})\right\|$. We have to show $\|(x_j, y_{-j})\| \geq \left\|(x'_j, y_{-j})\right\|$ for all y_{-j}. By Lemma 1, a monotonic norm is absolute. This implies:

$$\begin{aligned} \|(|x_1|, ..., |x_j|, ..., |x_d|)\| &= \|(x_1, ..., x_j, ..., x_d)\| \\ &\geq \left\|(x_1, ..., x'_j, ..., x_d)\right\| \\ &= \left\|(|x_1|, ..., \left|x'_j\right|, ..., |x_d|)\right\| \end{aligned}$$

Monotonicity implies then that $|x_j| \geq \left|x'_j\right|$. Hence, again by monotonicity, we obtain

$$\begin{aligned} \|(y_1, ..., x_j, ..., y_d)\| &= \|(|y_1|, ..., |x_j|, ..., |y_d|)\| \\ &\geq \left\|(|y_1|, ..., \left|x'_j\right|, ..., |y_d|)\right\| \\ &= \left\|(y_1, ..., x'_j, ..., y_d)\right\| \end{aligned}$$

as desired.

For the converse, assume that the preference induced by the norm $\|\cdot\|$ is separable. By Lemma 1, it is enough to show that the norm is absolute: for all $\mathbf{x} = (x_1, ..., x_d)$,

$$\|(x_1, ..., x_d)\| = \|(|x_1|, ..., |x_d|)\|.$$

If for all j, $x_j \geq 0$ or if for all j, $x_j \leq 0$, the implication is clear by the homogeneity of the norm. Assume then that $x_j \geq 0$ for all $j \in S$, and $x_j < 0$ for all $j \in S^C$, and that both S and S^C are not empty. Let $k \in S^C$ be minimal, and consider the vector $(x_1, ..., x_k, |x_{k+1}|, ..., |x_d|)$. Since k is minimal in S^C, we have

$$\|(x_1, ..., x_k, |x_{k+1}|, ..., |x_d|)\| = \|(|x_1|, ..., -|x_k|, |x_{k+1}|, ..., |x_d|)\|.$$

[25] Note that orthants are also defined by the chosen coordinate system.

We want to show that

$$\|(|x_1|, ..., -|x_k|, |x_{k+1}|, ..., |x_d|)\| = \|(|x_1|, ..., |x_k|, |x_{k+1}|, ..., |x_d|)\|.$$

Assume by contradiction that this is not the case, and let

$$\|(|x_1|, ..., -|x_k|, |x_{k+1}|, ..., |x_d|)\| < \|(|x_1|, ..., |x_k|, |x_{k+1}|, ..., |x_d|)\|. \tag{6}$$

The other case is completely analogous. By separability we obtain that

$$\|(-|x_1|, ..., -|x_k|, -|x_{k+1}|, ..., -|x_d|)\| < \|(-|x_1|, ..., |x_k|, -|x_{k+1}|, ..., -|x_d|)\|.$$

Multiplying the vectors by -1 and using homogeneity of the norm, we obtain

$$\|(|x_1|, ..., |x_k|, |x_{k+1}|, ..., |x_d|)\| < \|(|x_1|, ..., -|x_k|, |x_{k+1}|, ..., |x_d|)\|.$$

which is a contradiction to (6). Hence, we must have

$$\|(x_1, ..., x_k, |x_{k+1}|, ..., |x_d|)\| = \|(|x_1|, ..., |x_k|, |x_{k+1}|, ..., |x_d|)\|.$$

Continuing in the same way for all remaining $j \in S^C$ yields the desired result. ■

Remark 2 *Our discussion above stresses the dependence on the chosen coordinate system. This feature is not discussed at all by BGS, nor by Border and Jordan, but is crucial to our analysis. To understand the difference, note that both BGS and Border and Jordan implicitly fix a coordinate system. This also fixes the set of l_1 shortest paths BGS consider in the definition of m.s.p. Such paths are then solely composed of segments that are parallel to their fixed coordinates.*

In contrast to our present analysis, both BGS and Border and Jordan consider a rich (even maximal in the BGS analysis) class of preferences for which very particular marginal medians are DIC: all DIC mechanisms they find are separable in the sense that they can be decomposed into d one-dimensional DIC mechanisms (again, with respect to their fixed system). Non-separable mechanisms fail DIC because of some general preference in the respective rich domains. This also allows BGS, and Border and Jordan to prove converse statements about (separable) generalized medians.

*What we do here is different: we fix one particular preference relation generated by a norm (or, for some results below, a relatively "small" set of preferences, such as the preferences generated by all l_p norms) and look instead for **all** coordinate systems – each one defines then its own l_1 shortest paths and its own orthants – that yield DIC marginal medians for this particular preference. Thus, in each particular instance, we analyze a small set of preferences, and we are therefore able to uncover a larger set of DIC mechanisms. It is **not** the case that all these mechanisms are separable with respect to a given set of coordinates!*

4 Incentive Compatibility and Orthogonality

In mechanism design exercises one usually seeks to identify an optimal mechanism in a certain class of incentive compatible mechanisms. Thus we first need to identify the relevant incentive compatible mechanisms. How can we "construct" the issues to be voted upon that induce DIC marginal medians? In other words, given a norm, what are the coordinates that render this norm orthant-monotonic? This is a rather non-trivial problem whose solution requires several advanced techniques from functional analysis/geometry. We first discuss several important geometric insights towards answering this question, and in the next Section we complete the answer via an analytic device.

Let us start with an example showing that the Euclidean norm on the plane has an orthant-monotonic representation if and only if it is computed according to a coordinate system defined by an **orthogonal** basis. This relates of course to the well-known observation that, under the Euclidean norm, marginal medians on the plane are DIC if and only if they are computed with respect to an orthogonal set of coordinates (see Kim and Roush [1984] and Peters et al [1992]). This result about the Euclidean norm generalizes to more dimensions.

Example 3 *(Orthogonality) Let* $\{\mathbf{e}^1, \mathbf{e}^2\}$ *be the standard basis for* $\mathbb{R}^2$ *and consider the Euclidean* $l_2(2)$ *norm with respect to this basis. Consider another algebraic basis* $\left\{\mathbf{f}^1, \mathbf{f}^2\right\}$ *that can be written as* $\mathbf{f}^1 = a_1\mathbf{e}^1 + a_2\mathbf{e}^2$ *and* $\mathbf{f}^2 = b_1\mathbf{e}^1 + b_2\mathbf{e}^2$*, where the matrix*

$$A = \begin{pmatrix} a_1 & a_2 \\ b_1 & b_2 \end{pmatrix}$$

is non-singular. Then, any vector $\mathbf{x}$ *can be represented as:*

$$\begin{aligned} \mathbf{x} &= x_1\mathbf{f}^1 + x_2\mathbf{f}^2 \\ &= x_1(a_1\mathbf{e}^1 + a_2\mathbf{e}^2) + x_2(b_1\mathbf{e}^1 + b_2\mathbf{e}^2) \\ &= (x_1a_1 + x_2b_1)\mathbf{e}^1 + (x_1a_2 + x_2b_2)\mathbf{e}^2 \end{aligned}$$

Note that (x_1, x_2) *are here the coordinates of* $\mathbf{x}$ *according to basis* $\{\mathbf{f}^1, \mathbf{f}^2\}$*. By Lemma 1, the* $l_2(2)$ *norm (with respect to Cartesian coordinates) is orthant monotonic in the coordinate system defined by* $\{\mathbf{f}^1, \mathbf{f}^2\}$ *if and only if*

$$||(x_1, x_2)|| \geq \max\{||(0, x_2)||, ||(x_1, 0)||\}.$$

By the formula of the $l_2(2)$ *norm with respect to the standard Cartesian basis* $\{\mathbf{e}^1, \mathbf{e}^2\}$*, we obtain that*

$$\begin{aligned} ||(x_1, x_2)|| &= \sqrt{(x_1a_1 + x_2b_1)^2 + (x_1a_2 + x_2b_2)^2}, \\ ||(0, x_2)|| &= \sqrt{(x_2b_1)^2 + (x_2b_2)^2}, \\ ||(x_1, 0)|| &= \sqrt{(x_1a_1)^2 + (x_1a_2)^2}. \end{aligned}$$

Thus, orthant-monotonicity in the coordinate system defined by $\{\mathbf{f}^1, \mathbf{f}^2\}$ *holds if and only if, for any* x_1 *and* x_2 *we have:*

$$(x_1a_1 + x_2b_1)^2 + (x_1a_2 + x_2b_2)^2 \geq \max\left\{(x_2b_1)^2 + (x_2b_2)^2, (x_1a_1)^2 + (x_1a_2)^2\right\}$$

This is equivalent to:

$$x_1^2(a_1^2 + a_2^2) + 2x_1x_2(a_1b_1 + a_2b_2) \geq 0 \quad \text{and} \quad x_2^2(b_1^2 + b_2^2) + 2x_1x_2(a_1b_1 + a_2b_2) \geq 0.$$

The above inequalities hold for all x_1 *and* x_2 *if and only if* $a_1b_1 + a_2b_2 = 0$*. In other words, the representation of the* $l_2(2)$ *norm is orthant monotonic with respect to* $\{\mathbf{f}^1, \mathbf{f}^2\}$ *if and only if the inner-product of* $\mathbf{f}^1$ *and* $\mathbf{f}^2$ *is zero (i.e.,* $\mathbf{f}_1$ *and* $\mathbf{f}_2$ *are orthogonal). Every such system yields an orthant-monotonic representation of the* $l_2(2)$ *norm, and thus each marginal median computed with respect to any orthogonal basis is DIC. Since the median is not a linear function (while orthogonal basis transformations are), this operation potentially yields new DIC mechanisms.*

4.1 The Birkhoff-James Orthogonality Relation

The standard definition of orthogonality used above (via a zero *inner-product*) can only be applied to *Hilbert spaces* where the norm is indeed generated by an inner product, e.g., the Euclidean norm.

In order to characterize the system of coordinates that yield DIC marginal medians in arbitrary normed spaces, we need a more general notion of orthogonality, introduced by Birkhoff [1935] and later masterfully analyzed by James [1947].[26]

A vector $\mathbf{x}$ is said to be *Birkhoff-James (BJ) orthogonal* to another vector $\mathbf{y}$ if $\mathbf{x}$ has the smallest norm among all vectors on the line through $\mathbf{x}$ that is parallel to $\mathbf{y}$. Equivalently,

[26] There are many definitions of orthogonality that generalize the standard one. But only the BJ notion is relevant for incentive compatibility. Also from a purely mathematical point of view this seems the "right" generalization - see below the connection to semi-inner products and normality.

the line through $\mathbf{x}$ that is parallel to $\mathbf{y}$ is tangent to the "ball" with radius $\|\mathbf{x}\|$.

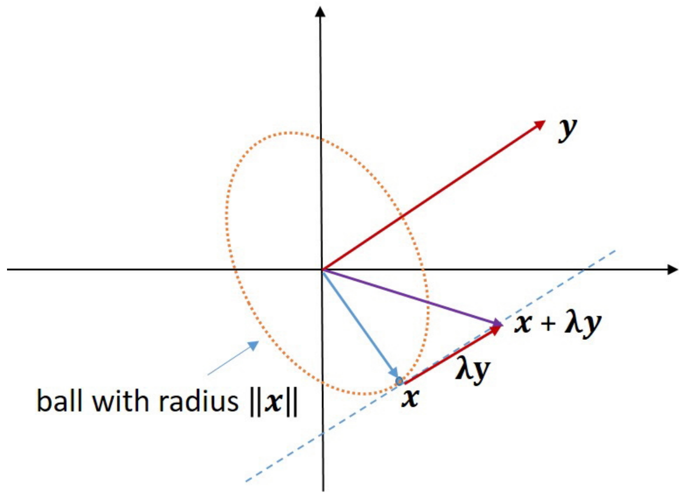

Figure 3: $\mathbf{x}$ is BJ-orthogonal to $\mathbf{y}$

Definition 5 *(Birkhoff-James orthogonality, Birkhoff [1935], James [1947]):*

1. *A vector* $\mathbf{x}$ *is orthogonal to another vector* $\mathbf{y}$*, denoted* $\mathbf{x} \dashv \mathbf{y}$*, if* $\|\mathbf{x} + \lambda\mathbf{y}\| \geq \|\mathbf{x}\|$ *for all real* λ*;* $\mathbf{y}$ *is orthogonal to* $\mathbf{x}$*, denoted* $\mathbf{y} \dashv \mathbf{x}$*, if* $\|\mathbf{y} + \lambda\mathbf{x}\| \geq \|\mathbf{y}\|$ *for all real* λ*.*

2. *We call* $\mathbf{x}$ *and* $\mathbf{y}$ *BJ-mutually orthogonal if* $\mathbf{x} \dashv \mathbf{y}$ *and* $\mathbf{y} \dashv \mathbf{x}$*.*

3. *A vector* $\mathbf{x}$ *is BJ-orthogonal to a subspace* M*, denoted* $\mathbf{x} \dashv M$*, if* $\mathbf{x} \dashv \mathbf{y}$ *for all* $\mathbf{y} \in M$*. A subspace* M *is BJ-orthogonal to a vector* $\mathbf{x}$*, denoted* $M \dashv \mathbf{x}$*, if* $\mathbf{y} \dashv \mathbf{x}$ *for all* $\mathbf{y} \in M$*.*

The BJ-orthogonality relation is **not** symmetric: $\mathbf{x}$ can be orthogonal to $\mathbf{y}$ but not vice-versa. BJ-orthogonality reduces to the standard (and symmetric) definition if the space is Hilbert: two vectors are orthogonal if and only if their inner-product is zero.

Another main "defect" of the BJ orthogonality relation is that it is **not** necessarily additive, neither on the left, nor on the right: $\mathbf{y} \dashv \mathbf{x}$, $\mathbf{z} \dashv \mathbf{x}$ need not imply $(\mathbf{y} + \mathbf{z}) \dashv \mathbf{x}$, and also $\mathbf{x} \dashv \mathbf{y}$, $\mathbf{x} \dashv \mathbf{z}$ need not imply $\mathbf{x} \dashv (\mathbf{y} + \mathbf{z})$.

The next concept "corrects" for symmetry and for additivity on the right. Given an algebraic basis $\{\mathbf{x}^1, \mathbf{x}^2, ..., \mathbf{x}^d\}$, denote by X^{-i} the subspace spanned by the vectors $\{\mathbf{x}^1, ..., \mathbf{x}^{i-1}, \mathbf{x}^{i+1}, ..., \mathbf{x}^d\}$.

Definition 6 *An Auerbach basis is an algebraic basis* $\{\mathbf{x}^1, \mathbf{x}^2, ..., \mathbf{x}^d\}$ *such that, for each* $j = 1, ..., d$, $\mathbf{x}^j \dashv X^{-j}$*. Such a basis is orthonormal if* $\left\|\mathbf{x}^j\right\| = 1,\ j = 1, 2, ..., d$*.*

Theorem 2 *In any normed space there exist at least* ***two*** *distinct Auerbach bases (see Day [1947], Taylor [1947]). Moreover, there is* ***no*** *isometry that transforms one of these bases into another, unless the space is Hilbert (see Plichko [1991]).*

To illustrate the beautiful existence argument, consider a two-dimensional normed space and its unit ball. Let Q be the quadrilateral with largest area inscribed in the unit ball. It exists by compactness. Then its main diagonals are *conjugate diameters* (see Heil and Krautwald [1969]). In particular, the diagonals are in the directions of two BJ-mutually orthogonal vectors. Analogously, let Q' be the quadrilateral with smallest area such that the four sides are all tangents to the unit ball. Again, it exists by compactness. Then the two lines such that each connects two tangency points across each other are also conjugate diameters in the directions of BJ-mutually orthogonal vectors. The two constructions always yield different pairs, unless the space is Hilbert (then there are an infinite number of different pairs).

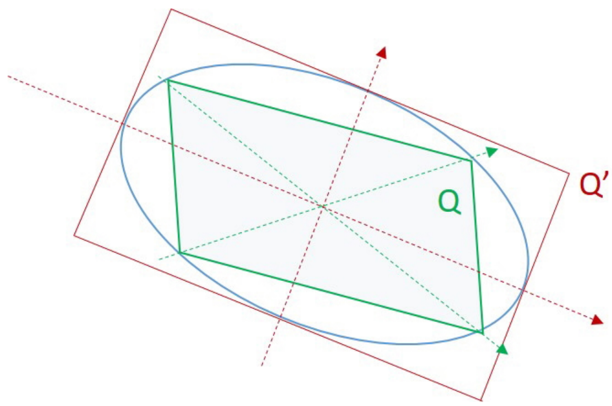

Figure 4: Inscribed parallelogram Q and circumscribing parallelogram Q'

Remark 3 *If $d > 2$, and if BJ-orthogonality is symmetric (i.e., $\mathbf{x} \dashv \mathbf{y} \Leftrightarrow \mathbf{y} \dashv \mathbf{x}$), then the space must be Hilbert. Surprisingly, there are norms on $\mathbb{R}^2$ defined by the so-called Radon curves that are not induced by an inner product, but that, nonetheless, have an infinite number of mutually orthogonal pairs and orthogonality is always symmetric. For an example, define the norm according to l_p for $\mathbf{x} = (x_1, x_2)$ such that $x_j x_k \geq 0$, but according to l_q, $1/p + 1/q = 1$ for $\mathbf{x} = (x_1, x_2)$ such that $x_j x_k < 0$. Note that this "composite" norm is orthant-monotonic because it combines monotonic norms orthant by orthant!*

4.2 Orthant-Monotonicity and Orthogonality

It turns out that the search for DIC mechanisms in higher-dimensional spaces does not reduce to the search for Auerbach bases. We show below that another property is crucial: in order to induce a DIC marginal median, for any j, any vector in the span of $\mathbf{x}^{-j}$ must be orthogonal to $\mathbf{x}^j$. Because of the lack of symmetry and additivity, this property does not automatically follows from the properties of mutually orthogonal vectors unless the space is Hilbert, where orthogonality is always symmetric and additive on both sides. Our next main result is:

Theorem 3 *Fix an algebraic basis* $\{\mathbf{x}^1, ..., \mathbf{x}^d\}$*. The norm* $||\cdot||$ *is orthant-monotonic with respect to the associated coordinate system if and only if*

$$X^{-j} \dashv \mathbf{x}^j \quad \text{for all } j = 1, ..., d. \tag{$*$}$$

Hence, a marginal median mechanism is DIC if and only if it is computed with respect to a basis $\{\mathbf{x}^1, \mathbf{x}^2, ..., \mathbf{x}^d\}$ *that satisfies property (*) above.*

Proof. The proof follows by the elegant geometric characterization of orthant-monotonic norms in Gries [1967] and Funderlic [1979]. Although those authors did not observe the relation to the BJ-orthogonality notion, their results are related to it.[27] We replicate here their method of proof, while translating it in terms of BJ-orthogonality.

Suppose first that the norm is orthant monotonic. Let X^{-j} denote the subspace spanned by $\left\{\mathbf{x}^1, ..., \mathbf{x}^{j-1}, \mathbf{x}^{j+1}, ..., \mathbf{x}^d\right\}$. If $\mathbf{z} \in X^{-j}$, then $\mathbf{z}$ and $\mathbf{z} + \beta\mathbf{x}^j$ lie in the same orthant for any β, since $z_k(z_k + \beta x_k^j) = z_k^2 \geq 0$ for all $k = 1, ..., d$. Moreover, $|z_k| \leq \left|z_k + \beta x_k^j\right|$ for all k. It follows from orthant monotonicity that

$$\|\mathbf{z}\| \leq \left\|\mathbf{z} + \beta\mathbf{x}^j\right\|, \quad \text{for all } \beta,$$

which means that $\mathbf{z} \dashv \mathbf{x}^j$.

Suppose now $X^{-j} \dashv \mathbf{x}^j$ for all j. Let $\mathbf{y}$ and $\mathbf{z}$ be in the same orthant such that $|z_j| \leq |y_j|$ for all $j = 1, ..., d$. We need to show that

$$\|(z_1, z_2, ..., z_d)\| \leq \|(y_1, y_2, ..., y_d)\|.$$

Let $\mathbf{v}^0 = \mathbf{z}$, $\mathbf{v}^d = \mathbf{y}$ and

$$\mathbf{v}^k = (y_1, y_2, ..., y_k, z_{k+1}, ..., z_d)^T, \ k = 1, 2, ..., d-1.$$

For the norm to be orthant monotonic, it is thus sufficient to show that

$$\left\|\mathbf{v}^{k-1}\right\| \leq \left\|\mathbf{v}^k\right\|, \ k = 1, 2, ..., d.$$

By the construction of $\mathbf{v}^k$ and because $\mathbf{y}$ and $\mathbf{z}$ are in the same orthant with $|z_j| \leq |y_j|$ for all j, there exists $\lambda \in [0, 1]$ such that

$$\mathbf{v}^{k-1} = \mathbf{v}^k - \lambda y_k \mathbf{x}^k = \lambda\left(\mathbf{v}^k - y_k\mathbf{x}^k\right) + (1-\lambda)\,\mathbf{v}^k.$$

By assumption and by the construction of $\mathbf{v}^k$, the vector $\mathbf{v}^k - y_k\mathbf{x}^k$ is in the subspace X^{-k} and is thus orthogonal to $\mathbf{x}^k$. Hence, we obtain

$$\left\|\mathbf{v}^k - y_k\mathbf{x}^k\right\| \leq \left\|\left(\mathbf{v}^k - y_k\mathbf{x}^k\right) + y_k\mathbf{x}^k\right\| = \left\|\mathbf{v}^k\right\|.$$

[27] Tanaka and Saito [2014] establish similar relations, but are not aware of the earlier papers by Gries and Funderlic.

It follows that

$$\begin{aligned}\left\|\mathbf{v}^{k-1}\right\| &= \left\|\lambda\left(\mathbf{v}^{k}-y_k\mathbf{x}^{k}\right)+(1-\lambda)\,\mathbf{v}^{k}\right\| \\ &\leq \lambda\left\|\mathbf{v}^{k}-y_k\mathbf{x}^{k}\right\|+(1-\lambda)\left\|\mathbf{v}^{k}\right\| \\ &\leq \left\|\mathbf{v}^{k}\right\|.\end{aligned}$$

This completes the proof. ■

Note that a necessary condition for property $(*)$ is that the vectors $\mathbf{x}^1, ..., \mathbf{x}^d$ in the basis $\left\{\mathbf{x}^1, ..., \mathbf{x}^d\right\}$ be mutually orthogonal. When $d = 2$, property $(*)$ is in fact equivalent to the mutual orthogonality of $\mathbf{x}^1$ and $\mathbf{x}^2$. Therefore, a two-dimensional marginal median computed with respect to the basis $\{\mathbf{x}^1, \mathbf{x}^2\}$ is DIC if and only if $\mathbf{x}^1$ and $\mathbf{x}^2$ are BJ-mutually orthogonal - this is shown in Peters et al [1993].

Example 4 *Let $\{\mathbf{e}^1, \mathbf{e}^2\}$ be the standard basis vectors in $\mathbb{R}^2$. Then the $l_p(2)$ norm according to this basis is monotonic. Now consider another basis $\{\mathbf{f}^1, \mathbf{f}^2\}$ with $\mathbf{f}^1 = (1,1)$ and $\mathbf{f}^2 = (-1,1)$. We will prove later in Section 6 that for any $l_p(2)$ norm, $\mathbf{e}^1$ and $\mathbf{e}^2$ are BJ-mutually orthogonal, and that $\mathbf{f}^1$ and $\mathbf{f}^2$ are also BJ- mutually orthogonal. For any vector $\mathbf{x}$ we can write*

$$\mathbf{x} = a\mathbf{f}^1 + b\mathbf{f}^2 = a(\mathbf{e}^1 + \mathbf{e}^2) + b(\mathbf{e}^2 - \mathbf{e}^1) = (a-b)\mathbf{e}^1 + (a+b)\mathbf{e}^2.$$

Hence, for the coordinates according to the basis $\{\mathbf{f}^1, \mathbf{f}^2\}$, the norm is computed as

$$||\mathbf{x}|| = ||a\mathbf{f}_1 + b\mathbf{f}_2|| = (|a-b|^p + |a+b|^p)^{1/p}.$$

To show orthant monotonicity we need

$$||a\mathbf{f}_1 + b\mathbf{f}_2|| \geq \max\{||a\mathbf{f}_1 + 0\mathbf{f}_2||, ||0\mathbf{f}_1 + b\mathbf{f}_2||\},$$

which is equivalent to

$$(|a-b|^p + |a+b|^p)^{1/p} \geq \max\left\{(2\,|a|^p)^{1/p}, (2\,|b|^p)^{1/p}\right\}.$$

Thus, it is enough to show that

$$|a-b|^p + |a+b|^p \geq \max\{2\,|a|^p, 2\,|b|^p\},$$

which holds by convexity. Hence, a marginal median with respect to the basis $\{\mathbf{f}^1, \mathbf{f}^2\}$ is DIC.

*Consider next a third basis $\{\mathbf{g}^1, \mathbf{g}^2\}$ with $\mathbf{g}^1 = (1,0)$ and $\mathbf{g}^2 = (1,1)$. We show below that $\mathbf{g}^1$ and $\mathbf{g}^2$ are **not** BJ-orthogonal. For any vector $\mathbf{x}$ we can write*

$$\mathbf{x} = a\mathbf{g}^1 + b\mathbf{g}^2 = (a+b)\mathbf{e}^1 + b\mathbf{e}^2.$$

Hence

$$||\mathbf{x}|| = \left\|a\mathbf{g}^1 + b\mathbf{g}^2\right\| = (|a+b|^p + |b|^p)^{1/p}.$$

Orthant monotonicity requires that

$$\begin{aligned}\left|\left|a\mathbf{g}^1 + b\mathbf{g}^2\right|\right| &\geq \left|\left|0\mathbf{g}^1 + b\mathbf{g}^2\right|\right| \\ &\Leftrightarrow \left(|a-b|^p + |b|^p\right)^{1/p} \geq \left(2\,|b|^p\right)^{1/p} \\ &\Leftrightarrow |a-b|^p \geq |b|^p\,.\end{aligned}$$

*The last inequality is false in general. Therefore, a marginal median with respect to this basis is **not** DIC.*

Remark 4 *One **cannot** require property (∗) to generally hold for any basis without excluding all cases of interest here. For $d > 2$, Marino and Pietramala [1987] proved the following: Let V be smooth, reflexive and strictly convex, and assume that for any triple $(\mathbf{x}, \mathbf{y}, \mathbf{z})$ of mutually orthogonal vectors it holds that $(\mathbf{x}+\mathbf{y}) \dashv \mathbf{z}$. Then V is a Hilbert space.*

5 How to Find DIC Marginal Medians?

Theorem 3 reduces the quest for DIC marginal medians to the quest for all bases consisting of BJ-mutually orthogonal vectors that satisfy property (∗). We first recapitulate how this is done for the Euclidean norm. Then we explain why this geometric procedure does not work for general normed spaces that are not Hilbert. Finally, we introduce an analytic approach based on *semi-inner products*, and show how it can be used to find all bases with property (*).

5.1 Isometries and Orthogonality

Finding orthogonal bases for the standard Euclidean norm is relatively straightforward:

1. Identify one orthogonal basis according to the standard Euclidean inner product.
2. Identify orientation preserving linear isometries (i.e., rotations) that are known to preserve orthogonality. These isometries yield additional orthogonal bases.
3. Any oriented orthogonal basis can be obtained (modulo translation) from any other via a suitable rotation.

Unfortunately, the property needed at step 3 does **not** hold in general normed spaces, i.e., *Euclid's fourth postulate* about the equivalence of right angles need not hold here. In addition, it is not even clear whether the set of maps that preserve BJ-orthogonality (step 2) is related to the set of linear isometries. The following elegant result shows that this is indeed the case:[28]

[28] This is related to a well known theorem about the linearity of surjective isometries, due to Mazur and Ulam [1932]

Theorem 4 *(Koldobsky, [1993]) Let V be a real normed space and let $T : V \to V$ be a linear operator preserving BJ-orthogonality, i.e., $\mathbf{x} \dashv \mathbf{y} \Rightarrow T(\mathbf{x}) \dashv T(\mathbf{y})$. Then $T = \lambda U$ where $\lambda \in \mathbb{R}$ and where U is an isometry.*

The usefulness of the above result for the purpose of finding all DIC mechanisms is somewhat diminished by the following surprising result: it starkly contrasts the analog result for inner product spaces, where the group of isometries is always a continuum.

Theorem 5 *(See e.g., Garcia-Roig [1997]) Let V be a finite-dimensional real normed space where the norm $||\cdot||$ is **not** generated by an inner product. Then, the sub-group of linear isometries with determinant $+1$ is finite.*[29]

The above results imply the following: assuming that we found one BJ-mutually orthogonal pair, we can then identify additional BJ mutually orthogonal pairs via isometries, but only a finite number of them at a time[30]. Moreover, from Theorem 2 we know that there may exist distinct pairs of mutually orthogonal vectors such that there is no isometry that transforms one pair into another.

5.2 Semi-Inner Products and Normality

Given the above difficulties with a purely geometric approach, we need an analytic way to capture BJ-orthogonality. For each two vectors $\mathbf{x}, \mathbf{y} \in \mathbb{R}^d$ define two real-valued functions on the real line

$$\begin{aligned} f_{\mathbf{x}}^{\mathbf{y}}(\lambda) &= ||\mathbf{x} + \lambda \mathbf{y}||, \\ f_{\mathbf{y}}^{\mathbf{x}}(\lambda) &= ||\mathbf{y} + \lambda \mathbf{x}||. \end{aligned}$$

By the convexity of norms, these functions are convex. The sub-differential ∂f of a convex function f at λ is the (compact and convex) set of supporting hyperplanes at λ.[31] It contains a unique element, the derivative, whenever the function is differentiable.

By definition, $\mathbf{x} \dashv \mathbf{y}$ if $\|\mathbf{x} + \lambda \mathbf{y}\| \geq \|\mathbf{x}\| = \|\mathbf{x} + 0\mathbf{y}\|$ for all real λ. In other words, if $\mathbf{x} \dashv \mathbf{y}$, then $\lambda = 0$ must be a minimum point of $f_{\mathbf{x}}^{\mathbf{y}}$. Analogously, $\mathbf{y} \dashv \mathbf{x}$ if $\|\mathbf{y} + \lambda \mathbf{x}\| \geq \|\mathbf{y}\|$ for all real λ which implies that $\lambda = 0$ must be a minimum point of the function $f_{\mathbf{y}}^{\mathbf{x}}$. It is well known that λ is a minimum of a convex function f if and only if $\mathbf{0} \in \partial f(\lambda)$. These considerations yield:

Lemma 2 *Two vectors $\mathbf{x}, \mathbf{y}$ are BJ-mutually orthogonal if and only if*

$$\mathbf{0} \in \partial f_{\mathbf{x}}^{\mathbf{y}}(0) \cap \partial f_{\mathbf{y}}^{\mathbf{x}}(0).$$

[29] A positive determinant means that the isometry preserves orientation.

[30] As we show below for the l_p case, isometries need not yield **any** additional DIC mechanisms because medians are covariant with respect to all of them. This never happens in Hilbert spaces.

[31] The simplest example is the absolute value function on the real line: the sub-differential at zero is the entire interval $[-1, 1]$, while at all other points it coincides with the derivative which is either 1 or -1.

In order to use these insights for an analytic description of coordinate systems that yield DIC marginal medians, we couple the above observations with the following concept:

Definition 7 *A semi-inner product (SIP) is a real-valued function* $[\cdot,\cdot]$ *defined on* $V \times V$ *with the following properties:*

1. $[\mathbf{x}+\mathbf{z},\mathbf{y}] = [\mathbf{x},\mathbf{y}] + [\mathbf{z},\mathbf{y}],\ \forall \mathbf{x},\mathbf{y},\mathbf{z} \in V$;
2. $[\lambda\mathbf{x},\mathbf{y}] = \lambda[\mathbf{x},\mathbf{y}],\ \forall \mathbf{x},\mathbf{y} \in V,\ \forall\lambda \in \mathbb{R}$;
3. $[\mathbf{x},\lambda\mathbf{y}] = \lambda[\mathbf{x},\mathbf{y}],\ \forall \mathbf{x},\mathbf{y} \in V,\ \forall\lambda \in \mathbb{R}$;
4. $[\mathbf{x},\mathbf{x}] \geq 0,\ \forall \mathbf{x} \in V$, and $[\mathbf{x},\mathbf{x}] = 0 \Rightarrow \mathbf{x} = 0$;
5. $|[\mathbf{x},\mathbf{y}]|^2 \leq [\mathbf{x},\mathbf{x}][\mathbf{y},\mathbf{y}]$, $\forall \mathbf{x},\mathbf{y} \in V$.

A Minkowski space may be simultaneously endowed with many different SIPs. An SIP is *consistent* with the norm of V if $[\mathbf{x},\mathbf{x}] = ||\mathbf{x}||^2$. The main difference to an inner-product is that the SIP need not be additive in the second variable, nor commutative.

Definition 8 *(Normality) Let* $[\cdot,\cdot]$ *be an SIP defined on* V*. Then* $\mathbf{x}$ *is normal to* $\mathbf{y}$ *if* $[\mathbf{y},\mathbf{x}] = 0$ *and* $\mathbf{y}$ *is normal to* $\mathbf{x}$ *if* $[\mathbf{x},\mathbf{y}] = 0$*.*

Note the **order** of the vectors in the above definition. This is important because the SIP is not commutative.

The following theorem links normality with BJ-orthogonality through the norm subdifferential. Norm smoothness requires that, for any vector $\mathbf{x}$ on the unit ball there is a unique tangent to the ball at $\mathbf{x}$. For example, the $l_1(2)$ norm is not smooth at the "corner" $\mathbf{x} = (0,1)$ and at its signed permutations.

Theorem 6 *(Giles, [1967]) Assume that the norm* $\|\cdot\|$ *is smooth, i.e., for all* $\mathbf{x},\mathbf{y} \in \mathbb{R}^d$, $\lim_{\lambda\to 0}\left(||\mathbf{x}+\lambda\mathbf{y}|| - ||\mathbf{x}||\right)/\lambda$ *exists.*[32] *Then there exists a unique SIP,* $[\cdot,\cdot]$*, that is consistent with the norm on* V*. Moreover, for any* $\mathbf{x},\mathbf{y} \neq \mathbf{0}$*,*

$$\frac{df_{\mathbf{x}}^{\mathbf{y}}(\lambda)}{d\lambda}|_{\lambda=0} = \frac{[\mathbf{y},\mathbf{x}]}{||\mathbf{x}||}.$$

Therefore, in a smooth normed space,

$$\mathbf{x} \dashv \mathbf{y} \Leftrightarrow [\mathbf{y},\mathbf{x}] = 0.$$

Remark 5 *If the space is smooth and if* $\{\mathbf{x}^1,\mathbf{x}^2,...,\mathbf{x}^d\}$ *are BJ-mutually orthogonal (or Giles normal in the sense of the SIP), then we have* $\mathbf{x}^j \dashv X^{-j}$ *for all* $j = 1,...,d$*. This property holds because of the left-additivity of the SIP (and hence of normality) in the first input, and because of the equivalence between normality and orthogonality in such spaces (*$\mathbf{x} \dashv \mathbf{y} \Leftrightarrow [\mathbf{y},\mathbf{x}] = 0$*). Thus, under smoothness, the additional requirement of an Auerbach basis (see Definition 6) is trivially satisfied, but property* $(*)$ *may not be.*

[32] This property is called Gateaux differentiability.

We conclude this section by the construction of a SIP consistent for a given norm. Fix a basis $\{\mathbf{x}^1, ..., \mathbf{x}^d\}$ in $\mathbb{R}^d$ and let

$$N(\mathbf{x}) = \|\mathbf{x}\|$$

denote the norm functional, and note that it is homogeneous of degree 1. Giles [1967] constructed a consistent SIP as follows:

$$[\mathbf{y}, \mathbf{x}] = \sum_{j=1}^{d} \sum_{k=1}^{d} \frac{1}{2} \left(\frac{\partial^2}{\partial x_j \partial x_k} N^2(\mathbf{x}) \right) x_j y_k. \tag{7}$$

We now derive an alternative representation that is more amenable for our analysis below.

By homogeneity of $N(\mathbf{x})$, we have

$$N(\mathbf{x}) = \sum_{j=1}^{d} N_{x_j}(\mathbf{x}) x_j \quad \text{and} \quad \sum_{j=1}^{d} N_{x_j x_k}(\mathbf{x}) x_j = 0 \text{ for all } k. \tag{8}$$

Note that

$$\frac{1}{2} \frac{\partial^2}{\partial x_j \partial x_k} N^2(\mathbf{x}) = N_{x_j}(\mathbf{x}) N_{x_k}(\mathbf{x}) + N_{x_j x_k}(\mathbf{x}) N(\mathbf{x}).$$

This observation, together with (8), yields that

$$\sum_{j=1}^{d} \frac{1}{2} \left(\frac{\partial^2}{\partial x_j \partial x_k} N^2(\mathbf{x}) \right) x_j = N(\mathbf{x}) N_{x_k}(\mathbf{x}).$$

Therefore, we can re-formulate Giles's construction of the consistent SIP as

$$[\mathbf{y}, \mathbf{x}] = \sum_{j=1}^{d} \sum_{k=1}^{d} \frac{1}{2} \left(\frac{\partial^2}{\partial x_j \partial x_k} N^2(\mathbf{x}) \right) x_j y_k = N(\mathbf{x}) \sum_{k=1}^{d} N_{x_k}(\mathbf{x}) y_k. \tag{9}$$

6 Illustrations

In this section we offer several illustrations and applications of the above concepts and insights.

6.1 The two-dimensional case

The two-dimensional case has been previously studied by Peters, van der Stel, and Storcken [1993]. As we observed earlier, a norm is then orthant monotonic with respect to a given basis if and only if this basis consists of a pair of BJ-mutually orthogonal vectors.

Definition 9 *A norm $||\cdot||$ is strictly convex if $||\lambda \mathbf{x} + (1-\lambda)\mathbf{y}|| < 1$ for all $\mathbf{x}, \mathbf{y}$ with $||\mathbf{x}|| = ||\mathbf{y}|| = 1$ and for all $\lambda \in (0,1)$.*

Theorem 7 *(Peters, van der Stel, and Storcken [1993]) Assume that the norm on* $\mathbb{R}^2$ *is strictly convex and let the number of agents be odd. A direct revelation mechanism is anonymous, Pareto-optimal and DIC if and only if it is a marginal median with respect to coordinates defined by a basis formed by two BJ mutually orthogonal vectors*[33]*.*

The following result combines the above insight with ours and yields an operational method to compute **all** DIC, anonymous and Pareto optimal mechanisms on the plane:

Corollary 1 *Let* $\mathbb{R}^2$ *be endowed with a smooth and strictly convex norm* $N(\cdot)$*. A direct revelation mechanism is anonymous, Pareto-optimal and DIC if and only if it is a marginal median with respect to an orthonormal Auerbach basis* $\{\mathbf{x}, \mathbf{y}\}$ *,* $\mathbf{x} = (x_1, x_2)$ *and* $\mathbf{y} = (y_1, y_2)$ *, that satisfies the following system of four equations in four unknowns:*

$$\frac{\partial N(\mathbf{x})}{\partial x_1} y_1 + \frac{\partial N(\mathbf{x})}{\partial x_2} y_2 = 0; \qquad (10)$$

$$\frac{\partial N(\mathbf{y})}{\partial y_1} x_1 + \frac{\partial N(\mathbf{y})}{\partial y_2} x_2 = 0; \qquad (11)$$

$$N(\mathbf{x}) = 1; \qquad (12)$$

$$N(\mathbf{y}) = 1. \qquad (13)$$

This system has at least ***two*** *pairs of distinct solutions.*

Proof. Since medians are translation-covariant, it is enough to characterize BJ-mutually orthogonal pairs $\mathbf{x}$ and $\mathbf{y}$ with $\|\mathbf{x}\| = \|\mathbf{y}\| = 1$. These are equations (12) and (13). By Theorems 3 and 6, we need to find all mutually normal pairs on the unit ball. It follows from (9) that mutually normal vectors must satisfy the system:

$$[\mathbf{y}, \mathbf{x}] = N(\mathbf{x}) \left(\frac{\partial N(\mathbf{x})}{\partial x_1} y_1 + \frac{\partial N(\mathbf{x})}{\partial x_2} y_2 \right) = 0,$$

$$[\mathbf{x}, \mathbf{y}] = N(\mathbf{y}) \left(\frac{\partial N(\mathbf{y})}{\partial y_1} x_1 + \frac{\partial N(\mathbf{y})}{\partial y_2} x_2 \right) = 0,$$

which reduce to equations (10) and (11) since $\mathbf{x}, \mathbf{y} \neq \mathbf{0}$. To conclude, finding all solution to equations (10)-(13) yields all pairs of directions (or the issues) for which DIC marginal medians can be constructed. There are at least two distinct solutions by Theorem 2). ■

In the Appendix we **directly** compute, for illustration, **all** solutions to the above equations for $l_p(2)$ spaces.

[33] A direct revelation mechanism is Pareto optimal if $\psi(\mathbf{t}_1, .., \mathbf{t}_n) \in conv(\mathbf{t}_1, .., \mathbf{t}_n)$ for all $\mathbf{t}_1, ..., \mathbf{t}_n \in \mathbb{R}^d$, where $conv(\mathbf{t}_1, ..., \mathbf{t}_n)$ is the convex hull of $(\mathbf{t}_1, .., \mathbf{t}_n)$

6.2 The $l_p(d)$ spaces

We consider now the class of $l_p(d)$ norms with respect to the standard Cartesian coordinates[34]:

$$\|\mathbf{x}\|_p = \left(\sum_{j=1}^{d} |x_j|^p \right)^{1/p}$$

and we characterize **all** DIC marginal medians.

6.2.1 The Isometries of l_p spaces

As observed above, the Hilbert space $l_2(d)$ admits a continuum of different orthogonal bases and, modulo translations, each one can be obtained from another by applying a suitable isometry that preserves orientation (i.e., rotations). For any other $l_p(d)$ space, $p \neq 2$, the set of isometries is finite because none of these norms are generated by inner products. The following result identifies the finite, multiplicative group of isometries.

Theorem 8 *(Li and Son [1994]) For any $l_p(d)$, $p \geq 1$, $p \neq 2$, the set of isometries is independent of p, and is represented by the set of signed permutation matrices, i.e. permutation matrices where some of the 1 entries are replaced by -1 entries.*

For example, for any $l_p(2)$, $p \geq 1, p \neq 2$, the set of linear isometries that preserve orientation (and have therefore determinant $+1$) is represented by the four matrices:

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

This set of matrices represent the easily visualized set of rotations with angles $\{0,\ \pi/2,\ \pi,\ 3\pi/2\}$ that leave invariant the l_1 unit ball in the plane (and, as the Theorem shows) also all other l_p unit balls.

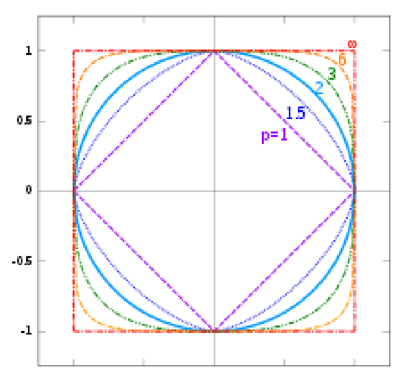

Figure 5: The l_p unit balls

[34] See Eguia[2011] for an axiomatic characterization of preferences derived form this class of norms.

Note that medians are covariant with respect to signed permutations with determinant $+1$, and hence after finding a BJ-mutually orthogonal pair, applying isometries does **not** reveal here new DIC mechanisms (in stark contrast to the Euclidean case). Moreover, we also know from the Theorem 2 that there is no isometry transforming one of BJ-mutually orthogonal pairs identified via maximal inscribed quadrilateral into the other, identified by minimum circumscribing quadrilateral. Therefore, we apply below the SIP method.

6.2.2 The Semi-Inner Product

Giles' construction of an SIP applied to the $l_p(d)$ norm - equivalently modified to best suit our purpose - is as follows. Let

$$N(\mathbf{x}) = \|\mathbf{x}\|_p = \left(\sum_{j=1}^{d} |x_j|^p\right)^{1/p}.$$

Suppose $x_j \neq 0$ for all j, and thus $d|x_j|/dx_j = x_j/|x_j|$. If $x_j = 0$, its impact on $[\mathbf{y},\mathbf{x}]_p$ vanishes in any case. Note that

$$\frac{\partial N(\mathbf{x})}{\partial x_j} = \frac{\partial}{\partial x_j}\left(\sum_{j=1}^{d} |x_j|^p\right)^{1/p} = \frac{1}{p}\left(\sum_{j=1}^{d} |x_j|^p\right)^{1/p-1} p\,|x_j|^{p-1}\frac{x_j}{|x_j|} = \frac{|x_j|^{p-2}x_j}{||\mathbf{x}||_p^{p-1}}$$

It follows from (9) that the SIP is given by

$$[\mathbf{y},\mathbf{x}]_p = N(\mathbf{x})\sum_{j=1}^{d} N_{x_j}(\mathbf{x})\,y_j = \sum_{j=1}^{d}\frac{|x_j|^{p-2}x_jy_j}{||\mathbf{x}||_p^{p-2}}. \tag{14}$$

This SIP is consistent with the $l_p(d)$ norm. Moreover, since the $l_p(d)$ norm, $p \neq 1$, is smooth, this is the unique SIP with this property. Finally, note that this formula coincides of course with the standard inner product formula for $p = 2$ where $[\mathbf{y},\mathbf{x}]_2 = \sum_{j=1}^{d} x_jy_j$.

6.2.3 The set of DIC marginal medians

We next characterize all Auerbach bases with property $(*)$[35], and hence all DIC marginal medians. We assume below that $p > 1$ because the "taxicab" (or "Manhattan") norm where $p = 1$ is not smooth. But, this case can be treated separately to yield the same general result. The case of $p = 2$, the standard Euclidean norm, is the unique one that is generated by an inner-product space. It admits a continuum of orthogonal bases, and we also exclude it from the analysis.

Theorem 9 *Fix $d \geq 2$ and assume that $p > 1$ and $p \neq 2$. For any $l_p(d)$ space, a marginal median is DIC if and only if it is computed with respect to coordinates defined by a modification of $E = \{\mathbf{e}^1, ..., \mathbf{e}^d\}$, the standard Cartesian basis obtained as follows:*

[35] By the smoothness of the norm, the right-additivity property is automatically satisfied here for $p > 1$.

1. *Choose a subset of vectors $E' \subseteq E$ with an even (possibly zero) number of elements, and partition it into distinct pairs $\{\mathbf{e}^j, \mathbf{e}^k\}$.*

2. *Replace each pair $\{\mathbf{e}^j, \mathbf{e}^k\}$ in E' by the pair $\{\mathbf{e}^j + \mathbf{e}^k, \mathbf{e}^j - \mathbf{e}^k\}$.*

3. *These new pairs, together with the remaining unit vectors in E/E', form a new basis.*

Proof. Using the SIP, Kinnunen [1984] proved the following: for any vector $\mathbf{x}$, there exists a hyperplane H such that $H \dashv \mathbf{x}$ if and only if $\mathbf{x}$ belongs to the union of the one-dimensional subspaces spanned $\mathbf{e}^j, \mathbf{e}^j + \mathbf{e}^k, \mathbf{e}^j - \mathbf{e}^k$, $j, k = 1, ..., d$, $j \neq k$. Kinnunen's result implies that each vector $\mathbf{x}^j$ in a basis with property $(*)$ must be one of the vectors in the statement (because, by property $(*)$, the hyperplane spanned by the other vectors in the basis is necessarily orthogonal to it).[36] Note that our construction yields indeed d linearly independent vectors. Kinnunen's result does not, however, determine the full composition of (Auerbach) bases with property $(*)$, and we do this below.

Let $\{\mathbf{x}^1, \mathbf{x}^2, ..., \mathbf{x}^d\}$ denote a basis as constructed in the statement. We first show that this basis satisfies property $(*)$. There are three cases to consider for a vector $\mathbf{x} \in \{\mathbf{x}^1, \mathbf{x}^2, ..., \mathbf{x}^d\}$:

1. $\mathbf{x} \in E \backslash E'$. Then $\mathbf{x} = \mathbf{e}^j$ for some j. Let $\mathbf{y}$ be a vector in the span of the other $d-1$ vectors in the basis, $\mathbf{y} = \sum_{k \neq j} \alpha^k \mathbf{e}^k$ for certain coefficients $\{\alpha^k\}$. We obtain from (14) that
$$[\mathbf{y}, \mathbf{x}]_p = \sum_{k=1}^{d} \frac{|x_k|^{p-2} x_k y_k}{||\mathbf{x}||_p^{p-2}} = 0,$$
because $x_k = e_k^j = 1$ for $k = j$, and $x_k = e_k^j = 0$ for $k \neq j$, and because $y_j = 0$.

2. $\mathbf{x} = \mathbf{e}^j + \mathbf{e}^k$. Let $\mathbf{y}$ be a vector in the span of the other $d-1$ vectors in the basis, $\mathbf{y} = \sum_{\ell \neq j,k} \alpha^\ell \mathbf{e}^\ell + \beta(\mathbf{e}^j - \mathbf{e}^k)$ for certain coefficients $\{\alpha^\ell\}$ and β. We obtain:
$$\begin{aligned} [\mathbf{y}, \mathbf{x}]_p &= \left[\sum_{\ell \neq j,k} \alpha^\ell \mathbf{e}^\ell + \beta(\mathbf{e}^j - \mathbf{e}^k), \mathbf{e}^j + \mathbf{e}^k\right]_p \\ &= \sum_{\ell \neq j,k} \alpha^\ell \left[\mathbf{e}^\ell, \mathbf{e}^j + \mathbf{e}^k\right]_p + \left[\mathbf{e}^j - \mathbf{e}^k, \mathbf{e}^j + \mathbf{e}^k\right]_p \\ &= \sum_{\ell \neq j,k} \alpha^\ell \left[\mathbf{e}^\ell, \mathbf{e}^j + \mathbf{e}^k\right]_p + \left[\mathbf{e}^j, \mathbf{e}^j + \mathbf{e}^k\right]_p - \left[\mathbf{e}^k, \mathbf{e}^j + \mathbf{e}^k\right]_p \\ &= 0 \end{aligned}$$
where the second and third equalities follow from the left-additivity of SIP in its first input, and the last equality follows because
$$\left[\mathbf{e}^\ell, \mathbf{e}^j + \mathbf{e}^k\right]_p = 0 \text{ for } \ell \neq j,k, \text{ and } \left[\mathbf{e}^j, \mathbf{e}^j + \mathbf{e}^k\right]_p = \left[\mathbf{e}^k, \mathbf{e}^j + \mathbf{e}^k\right]_p.$$

[36] A related result is in Lavric [1997]

3. $\mathbf{x} = \mathbf{e}^j - \mathbf{e}^k$. This case is analogous to case 2 above.

For the converse, we show that an (Auerbach) basis with property $(*)$ must have the form given in the statement of the Theorem. As implied by the result of Kinnunen [1984], each vector $\mathbf{x}^j$ in an Auerbach basis with property $(*)$ must be in the forms of $\mathbf{e}^j$, $\mathbf{e}^k - \mathbf{e}^\ell$ or $\mathbf{e}^m + \mathbf{e}^\ell$. Note that, by the left-additivity property of the SIP,

$$\left[\mathbf{e}^j - \mathbf{e}^k, \mathbf{e}^j\right]_p = \frac{1}{||\mathbf{e}^j - \mathbf{e}^k||_p^{p-2}} \neq 0 \text{ and } \left[\mathbf{e}^j + \mathbf{e}^k, \mathbf{e}^j\right]_p = \frac{1}{||\mathbf{e}^j + \mathbf{e}^k||_p^{p-2}} \neq 0.$$

Hence, if $\mathbf{e}^j$ belongs to an Auerbach basis, vectors of the form, $\mathbf{e}^k - \mathbf{e}^\ell$, $\mathbf{e}^k + \mathbf{e}^\ell$, $\mathbf{e}^\ell - \mathbf{e}^k$, can belong to the basis only if $k, \ell \neq j$.

If all vectors in the basis have their k-th coordinates equal to zero, then they cannot span the entire d dimensional vector space. Assume then that some vector of the form $\mathbf{e}^k + \mathbf{e}^j$ belongs to the basis. We need to show that $\mathbf{e}^k - \mathbf{e}^j$ also belongs to it (the proof of the opposite case is analogous). Assume by contradiction that $\mathbf{e}^k - \mathbf{e}^j$ does not belong to the basis. Since $\mathbf{e}^k$ cannot also belong to it by the preceding argument, the only other possible vectors in the basis that have a non-zero k-th coordinate are of the form $\mathbf{e}^k + \mathbf{e}^\ell$ or $\mathbf{e}^k - \mathbf{e}^\ell$ for $\ell \neq j$. But these vectors are not orthogonal to $\mathbf{e}^k + \mathbf{e}^j$ because, for all $\ell \neq j$,

$$\begin{aligned}
\left[\mathbf{e}^k + \mathbf{e}^j, \mathbf{e}^k + \mathbf{e}^\ell\right]_p &= \left[\mathbf{e}^k, \mathbf{e}^k + \mathbf{e}^\ell\right]_p + \left[\mathbf{e}^j, \mathbf{e}^k + \mathbf{e}^\ell\right]_p = \frac{1}{||\mathbf{e}^k + \mathbf{e}^\ell||_p^{p-2}} \neq 0, \\
\left[\mathbf{e}^k + \mathbf{e}^j, \mathbf{e}^k - \mathbf{e}^\ell\right]_p &= \left[\mathbf{e}^k, \mathbf{e}^k - \mathbf{e}^\ell\right]_p + \left[\mathbf{e}^j, \mathbf{e}^k - \mathbf{e}^\ell\right]_p = \frac{1}{||\mathbf{e}^k - \mathbf{e}^\ell||_p^{p-2}} \neq 0.
\end{aligned}$$

Therefore, $\mathbf{e}^k + \mathbf{e}^j$ and $\mathbf{e}^k + \mathbf{e}^\ell$ or $\mathbf{e}^k - \mathbf{e}^\ell$ for $\ell \neq j$ cannot simultaneously be part of an Auerbach basis. This completes the argument. ■

It is clear from Theorem 9 that the set of Auerbach bases with property $(*)$, and thus the set of DIC marginal median mechanisms, is **independent of** p. Theorem 9 tells us exactly how to find these DIC mechanisms for any l_p space. For example, for $d = 5$, Theorem 9 shows that the matrix of coordinates of the (column) vectors belonging to an Auerbach basis with property $(*)$ has either one of the forms below, or their signed permutations:

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & -1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & -1 \end{pmatrix}.$$

The case $d = 5$

Note that the number of distinct DIC marginal medians is larger for higher dimensions, which is somewhat surprising since one could expect that higher dimensions impose more constraints on incentive compatibility.

As an application, let us use the above result to identify **all** DIC, anonymous and Pareto-optimal mechanisms for the $l_p(2)$ spaces. The above Theorem shows that, modulo signed permutations, for all these spaces (with the exception of $p = 2$) there are exactly **two** pairs of directions on which marginal medians can be taken while preserving DIC: $\{\mathbf{e}^1, \mathbf{e}^2\}$ and $\{\mathbf{e}^1 + \mathbf{e}^2, \, \mathbf{e}^1 - \mathbf{e}^2\}$.[37] In particular, no matter what $l_p(2)$ norm an agent uses, reporting truthfully is DIC if a marginal median mechanism with respect to either one of these two bases is used. It is also clear that the two bases yield two distinct marginal median mechanisms: there is no isometry (i.e., there is no signed permutation) that transforms $\{\mathbf{e}^1, \mathbf{e}^2\}$ into $\{\mathbf{e}^1 + \mathbf{e}^2, \, \mathbf{e}^1 - \mathbf{e}^2\}$. The converse follows by Corollary 1.

A comparison of the two mechanisms from an utilitarian perspective (for the case $p = 2$) is conducted in Gershhkov, Moldovanu and Shi [2018]. There, we also establish the relations between theses two mechanisms and the *bottom-up* vs. *top-down* budgeting procedures used by legislatures (see for example Ferejohn and Krehbiel [1987], and Poterba and von Hagen [1999]).

6.3 Inner-Product Spaces

We briefly show here, for illustration, how our general approach also illuminates the often studied class of quadratic preferences: all such preferences correspond to norm-based preferences, where the norm is generated by an inner-product.

Assume first that agent i has an utility function derived from a weighted Euclidean norm with weights $\boldsymbol{\beta}_i \equiv (\beta_{i1}, \beta_{i2}, ..., \beta_{id}) \neq 0$ where $\beta_{ij} \geq 0$ for all i, j. That is, agent i with ideal point $\mathbf{t}_i = (t_{i1}, t_{i2}, ..., t_{id})$ has a utility from decision $\mathbf{x} = (x_1, ..., x_d)$ given by

$$-\sum_{j=1}^{d} \beta_{ij} \left(x_j - t_{ij}\right)^2 ,$$

Both $\mathbf{t}_i$ and $\boldsymbol{\beta}_i$ are agent i's private information. Let

$$A_i = \begin{pmatrix} \beta_{i1} & 0 & \cdots & 0 \\ 0 & \beta_{i2} & \cdots & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & \cdots & \beta_{id} \end{pmatrix},$$

and define an inner-product and its associated norm by:

$$\begin{aligned} \langle (x_1, ..., x_d), (y_1, ..., y_d) \rangle &\equiv (x_1, ..., x_d) A_i (y_1, ..., y_d)^T, \\ ||(x_1, ..., x_d)|| &\equiv \sqrt{(x_1, ..., x_d) A_i (x_1, ..., x_d)^T} = \sqrt{\sum_{j=1}^{d} \beta_{ij} x_j^2}. \end{aligned}$$

[37] These are, precisely, the conjugate diameters identified by Auerbach's theorem (Theorem 2) for these norms.

Each such inner-product is of course symmetric. In particular, the orthogonality relation is also symmetric. The unit ball is here the ellipse with axes parallel to the standard Cartesian coordinates, described by

$$\sum_{j=1}^{d} \beta_{ij} x_j^2 = 1,$$

We obtain the following "robust" mechanism design result:

Theorem 10 *Assume that there are at least d agents and that the set of possible weights determining their weighted Euclidean preferences contains d linearly independent vectors* $\boldsymbol{\beta}_1, ..., \boldsymbol{\beta}_d$*. Then, up to translations, the unique DIC marginal median is the one computed with respect to the standard Cartesian coordinates.*

Proof. Consider a realization where there are d agents such that each agent i has a utility function derived from a weighted Euclidean norms with weight vector $\boldsymbol{\beta}_i$. A DIC marginal median for such agents must be computed with respect to coordinates that are orthogonal under all norms generated by these various weights. Let us then look for a basis of the underlying vector space consisting of d vectors that are orthogonal from the joint point of view of all the d agents $i = 1, ..., d$. Property $(*)$ is satisfied here automatically since all norms are generated by inner-products.

Consider then d vectors, $\mathbf{x}^1, \mathbf{x}^2, ..., \mathbf{x}^d$, with $\mathbf{x}^k = (x_1^k, x_2^k, ..., x_d^k)$, $k = 1, ..., d$, all different from zero. Then any two vectors $\mathbf{x}^k, \mathbf{x}^\ell$ such that $\mathbf{x}^k \dashv \mathbf{x}^\ell$ must satisfy the following system of equations[38]:

$$\sum_{j=1}^{d} \beta_{ij} x_j^k x_j^\ell = 0, \text{ for all } i = 1, ..., d$$

Since $\boldsymbol{\beta}_1, ..., \boldsymbol{\beta}_d$ are linearly independent, we must have, for all $j, k, \ell \in \{1, ..., d\}$, $k \neq \ell$,

$$x_j^k x_j^\ell = 0.$$

This implies that, for each coordinate j, there is at most one k such that $x_j^k \neq 0$. As a result, there are at most d non-zero numbers in the set $\left\{x_j^k\right\}_{j,k=1,...,d}$. But since $\mathbf{x}^1, \mathbf{x}^2, ..., \mathbf{x}^d$ are all non-zero vectors, there is exactly one non-zero entry for each vector $\mathbf{x}^k$. In other words, the set $\{\mathbf{x}^1, \mathbf{x}^2, ..., \mathbf{x}^d\}$ consists of vectors in the direction of the standard Cartesian coordinates, and their signed permutations. The result follows because marginal medians are covariant with respect to signed permutations of the coordinates. ■

Border and Jordan [1983] studied DIC mechanisms on the entire class of *quadratic separable* preferences - these coincide with our weighted Euclidean norms described above. Combined with Moulin's characterization for one-dimensional domain, their Theorem 1 shows that any unanimous, anonymous and DIC mechanism must be a generalized median.

[38] This is, of course, a special case of the SIP approach described above.

As we showed above, these (generalized) medians must be computed with respect to the standard Cartesian coordinates.

Let us now consider what happens when we allow agents to have arbitrary (possibly different) utilities derived from inner-product norms. We recover then the impossibility result Theorem 3 of Border and Jordan [1983].

Theorem 11 *Assume that agents have individual preferences derived from arbitrary inner-product norms. Then there is no unanimous, anonymous and DIC mechanisms.*

Proof. Suppose there are two agents whose preference relations are generated from different weighted Euclidean norms. It follows from our result above, and from Theorem 1 in Border and Jordan [1983] that any unanimous, anonymous mechanism that is DIC for these two agents must be a generalized marginal median mechanism with respect to the standard Cartesian basis. Consider then a third agent whose preference relation is generated from a norm such that its associated unit ellipse is tilted with respect to the standard Cartesian coordinates. Then the Cartesian coordinates are **not** orthogonal according to this norm. To see this, assume that this third norm is generated by the following inner product

$$\langle (x_1, ..., x_d), (y_1, ..., y_d) \rangle \equiv (x_1, ..., x_d) A_i (y_1, ..., y_d)^T$$

where A_i is given by

$$A_i = \begin{pmatrix} 1 & b & \cdots & 0 \\ b & 1 & \cdots & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & \cdots & 1 \end{pmatrix}.$$

Since the unit ellipse defined by the norm is assumed to be tilted with respect to the Cartesian coordinates, we must have $b \neq 0$. Consider then the vectors

$$\mathbf{e}^1 = (1, 0, ..., 0) \text{ and } \mathbf{e}^2 = (0, 1, ..., 0).$$

We obtain

$$[\mathbf{e}^1, \mathbf{e}^2] = [\mathbf{e}^2, \mathbf{e}^1] = (1, 0, ..., 0) A_i (0, 1, ..., 0)^T = b \neq 0$$

and hence $\mathbf{e}^1$ and $\mathbf{e}^2$ are not orthogonal here. An MMMwith respect to the standard Cartesian coordinates is thus not DIC for such an agent, and the impossibility result follows. ■

7 Concluding Remarks

We have studied issue-by-issue voting by majority in a multi-dimensional collective decision situation, and we have identified all special systems of coordinates (the "issues") that render marginal median mechanisms incentive compatible. Our analysis has combined a variety of methods and concepts from geometry/functional analysis, a large part of which are novel to the Economics literature. For relatively small classes of norm-induced preferences we were able to construct incentive compatible mechanisms that cannot be otherwise identified when the class of feasible preferences is richer. Finally, by going well beyond the Euclidean distance function, our analysis opens a broader scope for applications of spatial voting analysis.

8 Appendix

We derive here - by direct computation via the SIP - all DIC, anonymous and Pareto-optimal mechanisms for the $l_p(2)$ spaces. By Corollary 1, all BJ-mutually orthonormal pairs are characterized by the system of four equations:

$$\begin{aligned} x_1y_1\,|y_1|^{p-2} + x_2y_2\,|y_2|^{p-2} &= 0, & (15)\\ y_1x_1\,|x_1|^{p-2} + y_2x_2\,|x_2|^{p-2} &= 0, & (16)\\ |x_1|^p + |x_2|^p &= 1, & (17)\\ |y_1|^p + |y_2|^p &= 1. & (18) \end{aligned}$$

Obvious solutions are the signed permutations of $(1, 0)$ and $(0, 1)$.

By Theorem 2, there must be at least another solution that cannot be obtained by isometry, i.e. cannot be obtained form the above by a signed permutation. .

From (15) and (16) either $x_1y_1 \leq 0$ and $x_2y_2 \geq 0$ or vice-versa, and $|x_j| \leq 1$, $|y_j| \leq 1$[39]. Assuming $x_2 < 0$ and $x_1, y_1, y_2 > 0$, we get:

$$\begin{aligned} x_1y_1^{p-1} + x_2y_2^{p-1} &= 0,\\ y_1x_1^{p-1} + x_2x_2^{p-1} &= 0,\\ x_1^p + x_2^p &= 1,\\ y_1^p + y_2^p &= 1. \end{aligned}$$

From the last two equations we get

$$x_1^{p-1} = (1 - x_2^p)^{1-1/p}\ ;\ y_1^{p-1} = (1 - y_2^p)^{1-1/p}\,.$$

Plugging these into the first two equations yields:

$$(1 - x_2^p)^{1/p}\,(1 - y_2^p)^{\frac{p-1}{p}} + x_2y_2^{p-1} = 0\ ;\ \ (1 - y_2^p)^{1/p}\,(1 - x_2^p)^{\frac{p-1}{p}} + y_2x_2^{p-1} = 0.$$

These are equivalent to:

$$(1 - x_2^p)\,(1 - y_2^p)^{p-1} = x_2^py_2^{p(p-1)}\ ;\ (1 - y_2^p)\,(1 - x_2^p)^{p-1} = y_2^px_2^{p(p-1)},$$

[39] Some care is needed for keeping the right signs; for example if $x_1, x_2, y_1 > 0$ then we necessarily have $y_2 < 0$.

which can be further rewritten as

$$\frac{1-x_2^p}{x_2^p} = \left(\frac{y_2^p}{1-y_2^p}\right)^{p-1} \quad ; \quad \frac{1-x_2^p}{x_2^p} = \left(\frac{y_2^p}{1-y_2^p}\right)^{1/(p-1)} .$$

We can use these two equations to obtain

$$\left(\frac{y_2^p}{1-y_2^p}\right)^{p-1} = \left(\frac{y_2^p}{1-y_2^p}\right)^{1/(p-1)} ,$$

which yields $y_2 = (1/2)^{1/p}$. Finally, it follows from equations (17) and (18) that $y_1 = x_1 = (1/2)^{1/p}$ and $x_2 = -(1/2)^{1/p}$. Thus we find the same two pairs of BJ-mutually orthogonal vectors for each $p > 1$_

$$\{\mathbf{e}^1, \mathbf{e}^2\} \text{ and } \{\mathbf{e}^1 + \mathbf{e}^2, \mathbf{e}^1 - \mathbf{e}^2\}$$

This is of course consistent with our general result (Theorem 9). The case $p = 1$ must be treated separately, but yields the same result.

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