A Theory of Auctions with Endogenous Valuations

Alex Gershkov, Benny Moldovanu and Philipp Strack*

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Abstract

We derive the symmetric, revenue maximizing allocation of m units among n symmetric agents who have unit demand, and who take costly actions that influence their values before participating in the mechanism. The auction with costly actions can be represented by a reduced form model where agents have convex, non-expected utility preferences over the interim probability of receiving an object. Both the uniform $m+1$ price auction and the discriminatory pay-your-bid auction with reserve prices constitute symmetric revenue maximizing mechanisms. Contrasting the case with exogenous valuations, the optimal reserve price reacts to both demand and supply, i.e., it depends both on the number of objects m and on number of agents n. The main tool in our analysis is an integral inequality involving majorization, super-modularity and convexity due to Fan and Lorentz (1954).

1 Introduction

In a variety of competitive environments, firms engage in pre-contract costly R&D activities. Tan (1992) documents an example where two groups of military contractors spent over \$600 million each **prior** to competitive bidding. In another well-known instance, bidders for spectrum licenses must both invest in infrastructure technology and retail networks in order

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to increase the value of their potential frequencies. Since network roll-over requirements to cover population and areas often impose strict schedules, a significant part of this investment – including the gathering of relevant business information that supports the commitment of large sums of money to pay for licenses - must be undertaken before bidding (see the discussion of ex-ante investments in the context of the recent FCC incentive auction in Milgrom (2017)).[1]

Decisions to invest depend on the nature of the required R&D and on its potential benefits (including those accruing in related areas of operation), the costs of R&D, the costs of preparing bids, and, last but not least, on the type of competitive contract-award procedure used by the designer. Designers of competitive allocation processes (e.g., the government in large, public procurement contracts or spectrum auctions) are well aware of the incentive effects of the mechanisms they choose, and do take various measures to encourage ex-ante investments: for example, the number of competitors in a bidding process may be a-priori restricted in order to ensure that each one of them has a reasonably high chance of winning, or the award is split among several winners, or reserve prices are adjusted downwards. On the other hand, such measures geared towards increasing ex-ante investments may have a detrimental effect on the obtained expected revenue at the bidding stage (or expected expenditure in procurement settings). This is the general trade-off we analyze here.

We derive the symmetric, revenue maximizing mechanism in a multi-unit auction framework where bidders undertake, prior to the auction, costly actions that are unobservable to the designer, and that influence their valuations at the subsequent auction. All other model's features are standard, and correspond to the symmetric, private independent values model with quasi-linear utility. The results immediately translate to procurement auctions where the designer acts as buyer. Besides some technical considerations (see below), the present focus on symmetric mechanisms is also justified by numerous practical applications that require the use of non-discriminatory mechanisms. For example, Article 18 (Principles of Procurement) of the EU's directive on public procurement states:

> Contracting authorities shall treat economic operators equally and without discrimination and shall act in a transparent and proportionate manner. The design of the procurement shall not be made with the intention of excluding it from the scope of this Directive or of artificially narrowing competition. Com-

[1]For example, incumbent potential winners at the 2019, forthcoming 5G spectrum auction in Germany are required to operate at least 1000 5G basis stations each until 2022, and to offer speeds of at least 100Mbit/second on all major rail-and highways until 2024. Newcomers must cover 25% of the population until 2023 and 50% until 2025 (See BK (2018)).

> petition shall be considered to be artificially narrowed where the design of the procurement is made with the intention of unduly favouring or disadvantaging certain economic operators.

Because of the ex-ante investments, the agents' valuations become endogenous to the mechanism. Hence, in contrast to the standard environment where the mechanism directly affects individual utilities via the physical allocation and the transfers, here a mechanism also needs to provide the right incentives for individual investments: thus, it also indirectly affects revenue through the *distribution of valuations*.

A revenue-maximizing seller can provide stronger ex-ante investment incentives by increasing the probability of allocating units, but is constrained by the limited supply and by the usual monopolistic supply reduction incentives. The revenue maximizing mechanism must finely balance these conflicting forces, and it needs to address a "fixed-point" problem that is not present in the standard model without investments: Although the revenue-maximizing mechanism for any fixed profile of investment strategies can be found by standard methods, the derivation of the overall optimal mechanism - including the optimal distribution of values that is induced by the mechanism itself - is more complex.

Myerson's (1981) celebrated analysis implies that, for every fixed distribution of investments (and thus for every fixed distribution of valuations), the seller's best reply is an auction with reserve price. This means that a seller who designs the mechanism after investments were undertaken will optimally select such an auction. But, to solve for the optimal mechanism in the case where the seller must select a mechanisms before agents invest, one needs to:

1. Characterize the endogenous distribution of valuations generated by the investments induced by each feasible mechanism.

2. Derive the expected revenue for each feasible mechanism, while taking into account the endogenous distribution of values it generates.

3. Maximize over feasible mechanisms.

As the endogenous distribution of values may be non-smooth and irregular, such a "naive" approach leads to serious technical challenges in sufficiently general settings. The principal technical innovation of the paper, that allows a treatment of considerable generality, is the solution via a novel approach that combines: 1) a reduced-form modeling where bidders have non-expected utility preferences that are **convex** in the interim probability of obtaining an unit, and 2) optimization based on the Fan-Lorentz (1954) integral inequality.

It is well known from the decision-theoretic literature (see, for example, Kreps and Porteus (1979)) that temporal preferences induced by actions that are taken prior to the resolution of uncertainty become non-linear and convex in probabilities - this is, precisely, the insight we exploit here by solving, instead of the original auction problem with investments, a reduced-form auction problem without investments, but with convex non-expected preferences.

The reduced form model presents it's own challenges: the main technical difficulty is then the non-linearity of ex-ante valuations in the allocation probability, which implies that the revenue maximization exercise cannot be performed "realization by realization". The direct subject of maximization becomes the interim probability of getting an unit - the "reduced form auction" in the language of Maskin and Riley (1984), Matthews (1984) and Border (1991). Therefore, besides the standard monotonicity constraint stemming from the incentive compatibility requirement, the most complex constraint in our setting is a resource constraint. In order to deal with it, we introduce a novel optimization technique that combines a characterization of reduced form, multi-unit auctions due to Che, Kim and Mierendorff (2013) with insights gained from majorization theory (pioneered by Hardy, Littlewood and Polya (1929)). The main analytical tool is an elegant integral inequality, due to Fan and Lorentz (1954), that combines *majorization*, *convexity* and *super-modularity.*

Under a super-modularity and convexity condition on the preferences, the optimization exercise yields an interim probability assignment such that units are allocated to the agents with the highest types, conditional on these exceeding a critical cutoff.[2] After deriving the optimal, symmetric direct revelation mechanism, we derive the bidding equilibria of the uniform $(m+1)-$ price auction and of the discriminatory pay-your-bid auction with a reserve price (where m is the number of supplied units), and show that these auctions, augmented by a reserve price, implement the revenue maximizing allocation.

Next, we offer comparative statics results with respect to the critical cutoff type below which no unit is allocated, and to the optimal reserve price, respectively. These results turn out to be very different from the classical ones obtained in the standard framework without investments. While in the setting with exogenous valuations the optimal reserve price coincides with the optimal cutoff, and neither is responsive to changes in demand or supply, here we find that reserve price and cut-off type are distinct, and often display

[2]The present optimization approach throws a new light also on the classical results obtained in the linear case, masterfully analyzed by Myerson (1981) and by Riley and Samuelson (1981) - their setting with exogenous valuations is a special case of our present framework.

comparative statics in opposite directions:

1. Under the same general conditions used for the maximization exercise described above, the optimal cutoff increases in the number of agents, and decreases in the number of objects.

2. Under some additional sufficient conditions, the optimal reserve price decreases in the number of agents and increases in the number of objects.

We also discuss several simple and practical methods to increase revenue, such as splitting the award, or ex-ante restricting the number of bidders.[3]

A variety of settings form special cases of our maximization framework and we offer illustrations to auctions where the endogenous values are influenced either additively or multiplicatively by costly actions, and to auctions with entry costs. Other setting such as crowd-sourcing contests can be solved by similar methods (see Online Appendix).

Finally, we briefly discuss the case where our super-modularity condition is not satisfied: we show how our focus on majorization yields a new angle to the characterization of the "ironed" extreme points that do satisfy monotonicity. Similarly to the classical result of Myerson (1981), the optimal symmetric mechanism in the "non-regular" case is given by an auction where intervals of types are pooled in order to ensure monotonicity of the allocation.

The paper is organized as follows: At the end of the present Section we briefly survey the relevant literature. In Section 2 we describe the auction model. In Section 3 we formulate the revenue maximization problem and the main tools towards its solution. In Section 4 we derive the revenue maximizing allocation in the "regular" case. The focus of Section 5 is on standard auction formats. We show that such formats implement the optimal allocation, and we explain how the optimal reserve price is affected by demand and supply. In Section 6 we display applications to specific economic settings where an auction is preceded by costly investments. We also illustrate how additional, practical measures of auction design, such as ex-ante restricting entry and splitting the award, can be incorporated in our setting. Section 7 briefly looks at ironing. Section 8 concludes. All proofs and some additional derivations are contained in two Appendices.

1.1 Literature Review

Expected utility preferences over allocations are linear in the probability of allocation, but temporal preferences induced by actions that are taken prior to the resolution of uncertainty become convex in this probability. Such preferences have been studied by, among others,

[3]As mentioned above, in projects where investments are massive, this number is sometimes cut down to two firms.

Kreps and Porteus (1979) and Machina (1984).[4] Single-object auctions with general non-expected utility preferences were analyzed by Karni and Safra (1989) and by Neilson (1984). These authors did not assume risk neutrality over transfers, and also studied the effect of risky prizes. These features - that are not present in our model since we assume quasi-linear utility - lead, for example, to non revenue-equivalence among standard auction formats.[5]

The paper closest in spirit to ours is Zhang (2017). She analyzes a single-object, two-agent setting with additive investments and with quadratic investment costs. She derives the optimal mechanism (that may be asymmetric) using the special structure of that setting, and she also offers a condition on the ex-ante distribution of types under which the unconstrained optimal mechanism is symmetric.

There is an older, substantial literature on auctions with investments, often formulated in the dual procurement setting where the designer acts as a buyer. In that literature, the mechanism is taken as given - often a standard auction format such as a first- or second-price auction – and an optimal mechanism-design analysis is not performed. Our main contribution with this respect to this literature is to show that standard auctions are in fact optimal in relatively general environments with investments. We review below several papers from this strand, many of them using the same timing as in the present model.

In an early paper, King et al. (1992) consider symmetric buyers that have no ex-ante private information, and whose realized types depend on investments made prior to the auction. Thus, the timing there is common with the one here, and the authors show that the second-price auction leads to efficient investments (only one buyer invests), while the first-price auction (i.e, its mixed-strategy symmetric equilibrium) yields a higher revenue. In a recent paper, Hatfield et al. (2018) assume a similar time structure and relate the ex-post efficient allocation to strategy-proofness and ex-ante incentives.

Piccione and Tan (1996) also consider agents without ex-ante private information, but, in their model, the effect of investments is stochastic. They consider two scenarios with respect to timing – the auction is announced either before or simultaneously with the investments – and focus on the implementability of the full information first-best outcome. Tan (1992) considers the same timing as here (mechanism is announced, investments are undertaken,

[4]Most of the following literature has been confined to one-person decision problems. Independently of such representations, non-linear, a-temporal preferences in the allocation have been suggested as a realistic representation of some human choices by Kahneman and Tversky (1979) in their famous Prospect Theory and Quiggin (1982). Consequences of multi-agent equilibrium behavior were first analyzed by Crawford (1990). A recent contribution is Dillenberger and Raymond (2017).

[5]Karni and Safra focus on dynamic inconsistency in a dynamic auction format. This phenomenon is due to the non-expected utility assumption.

bidding occurs) while focusing on the difference in investments induced by fixed mechanisms to procure a single object. In his model, investments lead to a stochastic individual cost that depend on a privately known parameter, and the number of participants is determined by a free entry condition. Bag (1997) studies a procurement model where firms invest in cost-reduction prior to bidding. As above, the procurer commits to a mechanism ahead of the investment decisions by firms. If the procurer can charge discriminatory entry fees, a second-price auction uniquely implements the first-best outcome. Without entry fees, the procurer may want to bid-discriminate between ex-ante identical firms in order to induce investment by a favored firm.

Arozamena and Cantillon (2004) investigate firms' incentives for cost reduction in the first- and second-price auctions with observable ex-ante investments (note that we assume here that investments are not observable). Both auction formats are shown to reinforce asymmetries among market participants.

Che and Gale (2003) study a procurement setting where firms are ranked in terms of cost, and invest in an innovation whose quality is observable to the buyer but not to other competitors. The buyer chooses the number of competitors and a menu of prizes, while firms offer a quality and demand a prize. The firm that offers the highest surplus (quality minus prize) wins the contest. The timing is similar to the one used here: the contest (within the class described above) is announced before investments take place. The authors show that letting the two most efficient innovators participate is optimal. In a contest model with stochastic outcomes of investments, Fullerton and McAfee (2009) suggest the use of interim prizes to finalists in a two-stage process. This is related to the advantages of split-awards in procurement, analyzed by Anton and Yao (1989) and by Gong et al. (2012).

Li et al. (2006) consider a procurement model with sub-modular cost functions (this is the dual of the super-modularity assumption on value functions in our framework), but in their paper the mechanism is announced simultaneously with investments. Li et al. study the Nash equilibrium of the game between designer and agents, and their main finding is the lack of pure strategy equilibria in investment.

Another related literature stream is the one on auctions with entry cost: entry can be seen as a binary investment decision with a fixed cost, and the optimal auction derivation is then more amenable to standard analysis. The settings of Celik and Yilankaya (2009) and Menezes and Monteiro (2000), for example, are special cases of our model because the entry decision is made by privately informed agents (see Section 6.4).

2 Auctions with Ex-Ante Investments

There are $m \geq 1$ identical and indivisible objects, and $n \geq m$ ex-ante symmetric bidders. Each bidder $i \in \{1, \ldots, n\} = N$ has a type $\theta_i \in \Theta = [\underline{\theta}, \overline{\theta}] \subseteq \mathbb{R}_+$ that is her private information, and demands at most one object. Types are distributed $I.I.D.$ according to a distribution $F : \Theta \to [0, 1]$, with positive density $f > 0$.

2.1 Actions and Preferences

Before participating in the mechanism, each agent i takes an action a_i from a compact set $A \subset \mathbb{R}$. The taken action is private information to the agent (and therefore unobservable to the designer). For example, before a large procurement auction, a producer may make an investment in new machines in order to produce in the future at a lower cost.

Depending on her type θ_i, agent i has preferences over her own action $a_i \in A$, her own allocation $x_i \in [0, 1]$ and her own monetary transfer y_i. We assume that these are standard expected utility preferences, quasi-linear in the monetary transfer and separable in the cost of an action $c(a_i) \geq 0$. The preference can be thus represented by a utility function of the form:

$$x_i \, v(a_i, \theta_i) - y_i - c(a_i),$$

where the value for a unit of the good, $v(a_i, \theta_i) \in \mathbb{R}_+$, is assumed to be increasing in her type θ_i and in her action a_i, super-modular and non-negative.[6] In addition we assume that v is Lipschitz-continuous in θ. Furthermore, we assume that there exists a costless action $0 \in A$ such that $c(0) = 0$.[7]

Remark. In reality, the agent's action may also have an effect outside the mechanism, e.g., an investment in a cost reducing technology produces additional benefits in other areas of operation. We can include such settings in our analysis by having $v(a_i, \theta_i)$ measure the effect of the costly action on the valuation *inside* the mechanism, and by having $c(a_i)$ measure the cost net of the benefits *outside* the mechanism.

[6] The assumption that the value is non-negative corresponds to free disposal.

[7] This is a normalization: we can always subtract $\min_{a \in A} c(a)$ from the agent's utility without changing her preferences over actions, allocations, or transfers.

2.2 The dual procurement model

We framed the model as a classical (sale) auction, but we wish to stress that this is mathematically equivalent to the standard model of procurement where the designer is a buyer and where the agents are sellers with costly production technologies. In a procurement setting seller i incurs cost of $\tilde{v}(a_i, \theta_i)$ if she produces an unit. This cost depends on her investment level before the procurement auction, a_i, and on her type θ_i. Higher investment levels or higher types lead to lower cost . The cost $\tilde{v}$ is assumed to be sub-modular which means that higher types reduce their cost more by investing.

Without loss of generality, assume that the buyer pays every winning seller the cost incurred by the highest cost buyer type under no investment, $\tilde{v}(0, \underline{\theta})$, and asks the buyer to pay back an amount t_i. Seller i's utility is thus given by the production cost, $x_i\tilde{v}(a_i, \theta_i)$, plus the transfer he receives, $x_i\tilde{v}(0, \underline{\theta}) - t_i$, minus his investment cost $c(a_i)$:

$$-x_i\tilde{v}(a_i, \theta_i) + x_i\tilde{v}(0, \underline{\theta}) - t_i - c(a_i) = x_i v(a_i, \theta_i) - t_i - c(a_i).$$

where we define $v(a_i, \theta_i) = \tilde{v}(0, \underline{\theta}) - \tilde{v}(a_i, \theta_i)$. Then, the value function v satisfies all assumptions made above, and therefore the optimal mechanism in the procurement problem can be determined from the optimal mechanism in the classical (sale) auction problem, and vice-versa.

2.3 Timing

The timing is as follows (c.f. Figure 1):

1. Each agent privately observes her type;
2. The designer (i.e., seller in the auction interpretation) commits to a mechanism;
3. Each agent decides whether to participate in the mechanism, and privately chooses an action;
4. Each agent sends a message to the mechanism;
5. Depending on the sent messages, an allocation and transfers are realized.

As documented in the introduction, this is the standard timing in the literature on procurement auctions for a designer acquiring multiple units from suppliers who may undertake costly investments, e.g., to reduce their supply cost. The assumed timing fits well

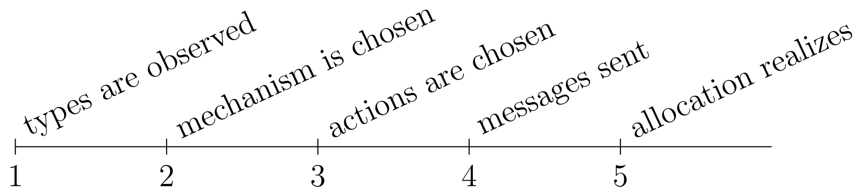

Figure 1: Timing of the game.

the procurement of technologically complex items (e.g., a battleship or a jet fighter) where a prototype (or a very detailed planning) is first required by the buyer. Building a prototype, say, requires an adjustment of existing technologies, and may lead to a future cost reduction, which is equivalent to an investment in the object's value in the standard auction setup.

2.4 Mechanisms

A *mechanism* specifies a set of reports R_i for each agent i, and a mapping from reports to an allocation and transfers:

$$\begin{aligned} x &: \prod_{i \in N} R_i \to X = [0,1]^n \\ y &: \prod_{i \in N} R_i \to \mathbb{R}^n \,. \end{aligned}$$

Given a mechanism (x, y) each agent i picks an optimal report r_i and takes an optimal action. As agent i can condition her action on the report she plans to send, her reporting problem is equivalent to

$$\max_{r_i \in R_i} \left(\left\{ \max_{a_i \in A} \mathbb{E}_{r_{-i}} \left[x_i(r) \right] \, v(a_i, \theta_i) - c(a_i) \right\} - \mathbb{E}_{r_{-i}} \left[y_i(r) \right] \right) \,. \tag{1}$$

We restrict attention to *symmetric mechanisms* that are invariant to permutations of the agents' names, and where every agent uses the same reporting strategy.

2.5 Reduction to Convex Preferences

We now show that an agent who takes an endogenous action obtains the same interim expected utility and sends the same report as an agent who takes no action, but has a non-expected utility preference. Define

$$p_i(r_i) := \mathbb{E}_{r_{-i}}\left[x_i(r)\right] \tag{2}$$

to be the interim probability with which agent i receives an object in a given mechanism. Let $h : [0,1] \times \Theta \to \mathbb{R}$ be the utility agent i gets when she receives an unit with probability p_i and takes the optimal action[8]

$$h(p_i, \theta_i) := \max_{a_i \in A} p_i \, v(a_i, \theta_i) - c(a_i) \, . \tag{3}$$

Then, the reporting problem of agent i given in (1) is equivalent to the reduced form problem where agent i has non-linear preferences over her allocation probability p_i

$$\max_{r_i \in R_i} h(p_i(r_i), \theta_i) - \mathbb{E}_{r_{-i}}\left[y_i(r)\right] \, .$$

Lemma 1. *$h(p_i, \theta_i)$ is convex in p_i, increasing in θ_i and p_i, super-modular in (p_i, θ_i) and satisfies $h(0, \theta_i) = 0$ for all $\theta_i \in \Theta$.*

The agent's benefit from receiving the object with higher probability is non-negative, and higher types derive higher utility from the object. Since h is defined independently of the agent's action, the mechanism design problem where the agent chooses an action before participating in the mechanism is equivalent to the mechanism design problem where each agent has a convex valuation given by $h_i(p_i, \theta_i)$, but chooses no costly action. Note also that the model with a potentially non-linear value function h (and hence with non-expected utility) includes as a special case the usual setup of auctions without costly actions, where the agent has expected utility preferences of the form

$$h(p_i, \theta_i) = p_i \cdot \theta_i \, .$$

3 The Revenue Maximization Problem

Consider direct mechanisms where each agent reports his type θ_i, and where the mechanism specifies allocations and monetary transfers to all agents, depending on the reported types.

[8]We assume that an optimal action always exists, which is satisfied for example if $\frac{\partial^2 v}{\partial a^2} \leq 0$ for all θ_i and $\frac{\partial^2 c}{\partial a^2} > 0$.

As $h(p_i, \theta_i)$ is super-modular, it is well-known that the incentive compatibility of a (symmetric) direct mechanism is equivalent to the standard monotonicity of the equilibrium expected allocation[9]

$$p(\theta_i) := \mathbb{E}_{\theta_{-i}}[x_i(\theta_1, ..., \theta_n)]$$

together with an envelope condition determining the expected interim transfer (see for example Guesnerie and Laffont (1984), Corollary 2.1). The envelope condition yields the following standard revenue formula:

Proposition 1 (Revenue Equivalence)**.** *The expected revenue in any symmetric, incentive compatible mechanism where the participation constraint is binding for the lowest type (i.e. the lowest type obtains zero utility) is given by*

$$n \int_{\Theta} H(p(\theta), \theta) \, f(\theta) \, d\theta \,, \tag{4}$$

where the "virtual utility" $H : [0,1] \times [\underline{\theta}, \overline{\theta}] \to \mathbb{R}$ is defined by

$$H(p, \theta) := h(p, \theta) - \frac{\partial h(p, \theta)}{\partial \theta} \times \frac{1 - F(\theta)}{f(\theta)} \,. \tag{5}$$

In order to simplify the notation, it will often be convenient to reformulate the problem in terms of the quantile $t = F(\theta)$ associated with a type θ.[10] For quantile t we define the probability $q(t) = p(F^{-1}(t))$ and the associated virtual utility

$$G(q, t) := H(q, F^{-1}(t)) \,. \tag{6}$$

As $\frac{d\theta}{dt} = \frac{1}{f \circ F^{-1}(t)}$, integration by substitution of (4) yields the following equivalent characterization of expected revenue.

Corollary 1. *The expected revenue in any symmetric, incentive compatible mechanism where the participation constraint is binding for the lowest type is given by*

$$n \int_0^1 G(q(t), t) dt \,. \tag{7}$$

Note, that the expression given in (7) equals (4) for the case where types are uniformly distributed on $[0, 1]$.

[9] As the mechanism is symmetric the definition of p is independent of the agent's identity i .

[10] Since F has a strictly positive density, F^{-1} is well defined.

3.1 The Resource Constraint

The non-linearity of the individual expected revenue G in the allocation probability q does not allow the use of a standard point-by-point maximization approach. The crucial constraint becomes the feasibility (or resource) constraint: what expected interim allocations functions can be obtained from feasible, of symmetric mechanisms? This question has been initially addressed (in the framework of a single object auction) by Matthews (1984), Maskin and Riley (1984), and Border (1991).[11] The following, more recent result characterizes the set of feasible interim allocation rules for multi-unit auctions with one-dimensional types:

Proposition 2 (Che, Kim & Mierendorff (2013), Corollary 4)**.** *Consider an auction with n bidders where m units are available. A symmetric, monotone interim allocation rule $q : [0,1] \to [0,1]$ is a reduced form of a feasible allocation rule where no agent obtains more than k objects if and only if, for each $t \in [0,1]$ it holds that*

$$n \int_t^1 q(s)ds \leq \sum_{i=0}^{n} \min\{i \cdot k, m\} \binom{n}{i} (1-t)^i \, t^{n-i} \, . \tag{8}$$

In order to use the above result, we present a novel Lemma that connects the Che, Kim & Mierendorff characterization to the concept of (weak) majorization:[12]

Lemma 2. *It holds that*

$$\sum_{i=0}^{n} \min\{i, m\} \binom{n}{i} t^{n-i} (1-t)^i = n \int_t^1 \phi_{m,n}(z) \mathrm{d}z$$

where $\phi_{m,n}(t)$ is the probability that at most $m-1$ out of $n-1$ agents have a type larger than the type associated with the quantile t :

$$\phi_{m,n}(t) := \sum_{i=0}^{m-1} \binom{n-1}{i} t^{n-1-i} (1-t)^i \, . \tag{9}$$

The function $\phi_{m,n}$ is increasing in t and m, and decreasing in n. Furthermore, $\int_0^1 \phi_{m,n}(t)dt = \frac{m}{n}$.

[11] These papers were written without connections to the earlier mathematical literature on the existence of measures with given marginals, e.g. Lorenz (1949), Gale (1957), Ryser (1957) and Strassen (1965).

[12] For a characterization of a single-object reduced form auction in terms of second-order stochastic dominance see Hart and Reny (2015).

Combining Proposition 2, Lemma 2 and Corollary 1, we can conclude that our revenue maximization problem is equivalent to the following problem:

Proposition 3 (Characterization of Revenue Maximizing Mechanisms)**.** *A symmetric mechanism is revenue maximizing if and only if the induced interim probability of receiving an object* $q(t)$ *solves*

$$\max_{q} n \int_0^1 G(q(t),t)\mathrm{d}t \tag{10}$$

subject to:

$$q(t) \in [0,1] \qquad \textit{for all } t \in [0,1] \tag{11}$$

$$q \textit{ non-decreasing} \tag{12}$$

$$\int_t^1 q(z)\mathrm{d}z \leq \int_t^1 \phi_{m,n}(z)\mathrm{d}z \qquad \textit{for all } t \in [0,1]\,, \tag{13}$$

where $t = F^{-1}(\theta)$*, where* G *is defined in (6) and where* $\phi_{m,n}$ *is defined in (9).*

3.2 Majorization and the Fan-Lorentz Inequality

In order to deal with the resource constraint identified above in (13), we first recall several concepts and results from the theory of majorization, pioneered by Hardy, Littlewood and Polya (1929).[13]

For non-decreasing $q, \bar{q} \in L^1(0,1)$ we say that $\bar{q}$ *majorizes* q, denoted by $q \prec \bar{q}$ if the following two conditions hold:

$$\begin{aligned} \int_t^1 q(v)dv &\leq \int_t^1 \bar{q}(v)dv \quad \text{for all } t \\ \int_0^1 q(t)dt &= \int_0^1 \bar{q}(t)dt. \end{aligned} \tag{14}$$

We say that $\bar{q}$ *weakly majorizes* q, denoted by $q \prec_w \bar{q}$ if the first condition above holds (but not necessarily the second). If $q \prec_w \bar{q}$ it is easily seen that there exists $q' \leq \bar{q}$ such that $q \prec q'$.

The above definitions can also be applied to any (possibly non-monotonic) functions

[13] The opposite relation has been popularized in the Economics literature under the name *second-order stochastic dominance*. As shown above, a symmetric allocation is feasible if it is less dispersed than the efficient allocation in this sense.

$q, \bar{q} \in L^1(0,1)$ if $q, \bar{q}$ in (14) are replaced by their non-decreasing re-arrangements. The set

$$\Omega(\bar{q}) := \{q : q \prec \bar{q}\}$$

called the *orbit* of $\bar{q}$, is weakly compact and convex. In particular, by Bauer's Maximum Principle (1958), a continuous, convex functional on $\Omega(\bar{q})$ attains its maximum on an extreme point of $\Omega(\bar{q})$.[14] The next elegant result, due to Ky Fan and G.G. Lorentz (1954) identifies a set of convex functionals such that **all** of them attain their maximum on $\Omega(\bar{q})$ precisely at $\bar{q}$.

Proposition 4 (Fan-Lorentz Theorem)**.** *Let $L : [0,1] \times [0,1] \to \mathbb{R}$ be such that $L(q,t)$ is convex in q and super-modular in (q,t). Let $q, \bar{q} : [0,1] \to [0,1]$ be two non-decreasing functions such that $q \prec \bar{q}$. Then*

$$\int_0^1 L(q(t),t)dt \leq \int_0^1 L(\bar{q}(t),t)dt \,.$$

Proposition 3 shows that an allocation is feasible if and only if it is weakly majorized by $\phi_{m,n}$. In order to use the Fan-Lorentz Theorem, we introduce a regularity assumption ensuring that their conditions are satisfied by our expected revenue functional:

Definition 1 (Convex-Supermodularity)**.** *We say that the environment is "convex super-modular" (CSM) if the virtual utility $H(p,\theta)$ defined in (5) is convex in p and super-modular in (p,θ).*

It is instructive to consider the implication of the above conditions in the standard model with linear preferences (Myerson, (1981)) where

$$H(p,\theta) = p\left(\theta - \frac{1-F(\theta)}{f(\theta)}\right) .$$

As H is linear in p, it is (weakly) convex in p. It is super-modular in (p,θ) if and only if the standard virtual value

$$J(\theta) = \theta - \frac{1-F(\theta)}{f(\theta)}$$

is non-decreasing. Thus, for the special case of linear preferences, our definition of a CSM environment reduces to the definition of a "regular" environment in Myerson (1981).

[14] Ryff (1967) has shown that $q \in \Omega(\bar{q})$ is an extreme point of this set if and only if $q = \bar{q} \circ \Psi$ where Ψ is a measure preserving transformation of $[0,1]$ into itself. This is analogous to the discrete case, where the above result is a corollary of the famous Birkoff-von Neumann Theorem.

In the Online Appendix we give sufficient conditions on the valuation function $h(p,\theta)$ and on the distribution function F so that the generalized virtual utility function $H(p,\theta)$ satisfies CSM. The treatment is similar to the one in Guesnerie and Laffont (1984) and Fudenberg and Tirole (1991) (Chapter 7) who studied the one-person contracting setting with a valuation function that is **concave** in the allocation.[15]

4 The Optimal Mechanism

We now derive the optimal symmetric mechanism using the Fan-Lorentz Theorem. To simplify notation we denote by

$$\psi_{m,n}(\theta) := \phi_{m,n}(F(\theta)) \tag{15}$$

the interim probability with which an agent of type θ receives an object when m objects are allocated efficiently among n agents. We also denote by $a_{m,n}(\theta)$ the optimal action for type θ under allocation ψ :

$$a_{m,n}(\theta) \in \arg\max_{a\in A} \psi_{m,n}(\theta)\, v(a,\theta) - c(a)\,.$$

Theorem 1 (Revenue Maximizing Allocation)**.** *Suppose the environment is convex super-modular. Then, the symmetric revenue maximizing mechanism allocates the m objects to the agents with the highest types, conditional on these exceeding a threshold $\theta^*_{m,n} \equiv \theta^*$, where θ^* is the unique solution to $H(\psi_{m,n}(\theta^*),\theta^*) = 0$, or equivalently,*

$$\psi_{m,n}(\theta^*)v(a_{m,n}(\theta^*),\theta^*) - c(a_{m,n}(\theta^*)) - \frac{1-F(\theta^*)}{f(\theta^*)}\psi_{m,n}(\theta^*)\left(\frac{\partial v(a_{m,n}(\theta^*),\theta^*)}{\partial\theta}\right) = 0$$

whenever this exists.[16] *In the optimal mechanism, the expected utility of the lowest type, $\underline{\theta}$ is zero.*

The intuition for the proof of Theorem 1 (see Appendix) is as follows. First, we argue that in a CSM environment the virtual value $G(q,t)$, expressed as a function of the quantile t, is convex and super-modular in (q,t) (Lemma 5). As G is convex in q, the functional $q \mapsto \int_0^1 G(q(t),t)dt$ is convex, and it therefore attains a maximum on an extreme point of

[15]In the one person case the resource constraint is not an issue.

[16]If $H(\psi_{m,n}(\theta),\theta) > 0$ for any $\theta \in [\underline{\theta},\overline{\theta}]$ we set $\theta^* = \underline{\theta}$, while if $H(\psi_{m,n}(\theta),\theta) < 0$ for any $\theta \in [\underline{\theta},\overline{\theta}]$ we set $\theta^* = \bar{\theta}$.

the set of monotone functions that satisfy the constraint (13). This constraint says that q is weakly majorized by the allocation probability $\phi_{m,n}$ of being among the m highest types. In turn, this implies that q is majorized by a function of the form $t \mapsto 1_{\{t \geq t^*\}} \times \phi_{m,n}(t)$. Hence, by the Fan-Lorentz inequality, an interim probability of the form $t \mapsto 1_{\{t \geq t^*\}} \times \phi_{m,n}(t)$ maximizes revenue. The optimality of the cutoff type $\theta^*_{m,n} \equiv \theta^*$ is proven by arguing that the expected revenue is quasi-concave in the cutoff θ^*, and that $H(\psi_{m,n}(\theta^*), \theta^*) = 0$ is the relevant first-order condition.

To better understand the intuition behind the optimality of allocating the units efficiently above a cut-off, let us compare the auction with investments to the standard case without investments: Here the probability of getting a unit affects the investment incentives, and hence indirectly influences the virtual value of every agent. The increase in virtual value (due to an increase in the probability of obtaining an unit) is more substantial for an agent with a higher type. Compared to the case where values are exogenous, this increases the advantage of allocating the units to agents with higher types, conditional on their virtual values (which depend on the allocation probability) being non-negative. It is thus not revenue maximizing to give the object to a lower type (i.e., withhold it from a higher type).

4.1 Comparative Statics

We provide some comparative statics of the optimal cutoff type with respect to the number of agents n and the number of objects m. In sharp contrast to the standard auction setting without investments, we show that the optimal cutoff type is influenced both by the number of agents (demand) and objects (supply).

From the definition of $\psi_{m,n}(\theta) = \phi_{m,n}(F(\theta))$, we know that $\psi_{m,n}(\theta)$ is increasing in m and θ and decreasing in n - these three basic properties account for our main result here:

Proposition 5 (Comparative Statics for the Cutoff Type). *Assume that the environment is convex super-modular (CSM). Then the optimal cutoff type $\theta^*_{m,n} \equiv \theta^*$ increases in the number of agents n and decreases in the number of objects m.*

The intuition behind Proposition 5 is as follows: An increase in the number of agents decreases the individual chance of each agent to receive an object. This, in turn, reduces the individual incentives to invest, the resulting values, and the revenue that can be obtained from each type. In particular, the type with zero virtual utility - that determines the optimal cutoff type θ^* - must be larger than the analogous type when there are less bidders.[17] As

[17] An inverse effect is at work when the number of objects increases.

they never receive an object, agents with types below θ^* choose the action with zero cost. The action chosen by higher types $\theta > \theta^*$ can be determined by maximizing their interim utility (given the interim probability with which they receive an object)

5 Implementation via Standard Auctions

In this section we analyze two widely used, standard auction formats that implement the revenue maximizing allocation in the model with costly investments.

5.1 The Uniform Price Auction

We first look at the $(m+1)-$ uniform price auction with a reserve price $\mathcal{R}$. Recall that $a_{m,n}$ is the optimal action in a $(m+1)$-price auction *without* reserve price

$$a_{m,n}(\theta) \in \arg\max_{a \in A} \psi_{m,n}(\theta)\, v(a,\theta) - c(a)\,.$$

The equilibrium strategy in the uniform price auction with n bidders, m objects and reserve price $\mathcal{R}$ is straightforward: agents with type above a cut-off type $\theta_{m,n,\mathcal{R}}$ participate in the auction by making a non-zero bid. Every type of agent who participates chooses an optimal action $a_{m,n,\mathcal{R}}$ that is equal to the action $a_{m,n}(\theta)$ this type would have chosen in a $(m+1)$-price auction without reserve:

$$a_{m,n,\mathcal{R}}(\theta) = \begin{cases} a_{m,n}(\theta) & \text{if } \theta \geq \theta_{m,n,\mathcal{R}} \\ 0 & \text{else} \end{cases}. \tag{16}$$

The critical type $\theta_{m,n,\mathcal{R}}$ – the lowest type who makes a non-zero bid – is indifferent between participating and not participating in the auction, i.e. his investment cost equals the expected gain from potentially winning an unit in the auction. This types solves:

$$\psi_{m,n}(\theta_{m,n,\mathcal{R}})\left[v(a_{m,n}(\theta_{m,n,\mathcal{R}}), \theta_{m,n,\mathcal{R}}) - \mathcal{R}\right] = c\left(a_{m,n}(\theta_{m,n,\mathcal{R}})\right). \tag{17}$$

The bid submitted by each participating type $\theta > \theta_{m,n,\mathcal{R}}$ is equal to the post-investment value from winning the auction, $v(a_{m,n}(\theta), \theta)$.

This profile of strategies (actions and bids) constitutes a standard Bayes-Nash equilibrium of the two-stage investment and bidding game.[18] In contrast to the case without investment where the bid is constant, here the equilibrium bid responds both to demand (the number of bidders) and supply (the number of objects) via the investment decision $a_{m,n,\mathcal{R}}$. More precisely, the equilibrium bid $v(a_{m,n,\mathcal{R}}(\theta), \theta)$ increases in θ and in the number of objects m, and decreases in the number of bidders n.

Remark. The equilibrium above displays a "jump" in the bidding function at the reserve price: the type $\theta_{m,n,\mathcal{R}}$ given in (17) submits a bid that exceeds the reserve price $\mathcal{R}$, or differently put, there exists an interval of types $[\theta_1, \theta_2]$ such that their value when investing optimally exceeds the reserve price $v(a_{m,n,\mathcal{R}}(\theta), \theta) > \mathcal{R}$, yet these types refrain from bidding.

For a type with post-investment value just above the reserve price, it is not profitable to invest and participate at the auction: in the best scenario where his type is among the m highest, he gets close to zero utility from winning, but looses the initial investment which leads to an overall loss.

Corollary 2. *Let $\theta^* \equiv \theta^*_{m,n}$ be the optimal cutoff defined in Theorem 1, and let $\mathcal{R}^* \equiv \mathcal{R}^*_{m,n,\theta^*}$ solve*

$$\mathcal{R}^* = \frac{1 - F(\theta^*)}{f(\theta^*)} \left(\frac{\partial v(a_{m,n}(\theta^*), \theta^*)}{\partial \theta} \right). \tag{18}$$

Then the uniform price auction with reserve price $\mathcal{R}^*$ is a symmetric revenue maximizing mechanism.

5.2 The Pay-Your-Bid Auction

We now show that the discriminatory Pay-Your-Bid Auction with a properly chosen reserve price also implements the revenue maximizing allocation. To do so, we first explicitly derive the equilibrium bidding strategies for the reduced-form auction with convex preferences.[19] This derivation is of independent interest since it covers cases where bidders do not have expected utility preferences. In the discriminatory Pay-Your-Bid Auction with reserve price, the m agents that submit the highest bids (conditional on these bids being above the reserve price) get the objects and pay their bids. Other agents pay nothing.

Proposition 6 (Equilibrium of the Discriminatory Auction)**.**

[18] This profile is **not** in dominant strategies: conditional on the other bids, an agent knows whether he wins or not and may prefer to change his investment decision.

[19] An alternative derivation can be obtained by a payoff equivalence argument.

1. *Assume that the seller uses a Pay-Your-Bid Auction with reserve price* $\mathcal{R}$, *and let* $\theta_{m,n,\mathcal{R}}$ *and* $a_{m,n,\mathcal{R}}(\theta)$ *be as defined above for the uniform price auction (see equations 16 and 17). Then, the profile of actions* $a_{m,n,\mathcal{R}}(\theta)$ *and bidding strategies*

$$\beta(\theta) = \begin{cases} \mathcal{R}\frac{\psi_{m,n}(\theta_{m,n,R})}{\psi_{m,n}(\theta)} + \frac{1}{\psi_{m,n}(\theta)}\int_{\theta_{m,n,R}}^{\theta_i} \frac{\partial\psi_{m,n}(z)}{\partial\theta} v(a_{m,n}(z), z)dz & \theta \geq \theta_{m,n,\mathcal{R}} \\ 0 & \theta < \theta_{m,n,\mathcal{R}} \end{cases}$$

constitute a symmetric, pure-strategy Bayes-Nash equilibrium in the Pay-Your-Bid Auction among n *bidders for* m *units.*

2. *The Pay-Your-Bid Auction with the reserve price that solves (18) constitutes a symmetric revenue maximizing mechanism.*

5.3 Comparative Statics: The Optimal Reserve Price

Both the Uniform Price and the Pay-Your-Bid auctions apply the same optimal reserve price. Proposition 5 gives a clear-cut result regarding the effect of the change in the number of objects and agents on the optimal cutoff type. It is important to note that the above result and intuition **do not** carry over to the optimal reserve price ! To see this, recall that

$$\mathcal{R}^*_{m,n,\theta^*} \equiv \mathcal{R}^* = \frac{h\left(\psi_{m,n}(\theta^*_{m,n}), \theta^*_{m,n}\right)}{\psi_{m,n}(\theta^*_{m,n})}.$$

There are two opposite effects at work here: on the one hand, from Proposition 5 it follows that an increase in the number of agents increases the cutoff type $\theta^*_{m,n}$; but, on the other hand, it also decreases the probability of getting the object of the cutoff quantile type, $\psi_{m,n}(\theta^*_{m,n})$. Since $\frac{h(p,\theta)}{p}$ is increasing in θ and increasing in p (due to convexity) the overall effect of an increase in the number of agent on the optimal reserve price is ambiguous. Differently put, the optimal reserve price makes the optimal cut-off type exactly indifferent between participating and not participating: an increase in the number of agents increases the cutoff type, but on the other hand, this type must be compensated for the increase in competition by reducing the reserve price. The next proposition gives sufficient conditions for the second effect to dominate. Inverse effects are at work for the dependence on the number of objects m.

Proposition 7 (Comparative Statics of the Optimal Reserve Price)**.** *Assume that the environment is convex super-modular (CSM). Assume further that* h *is concave in* θ *and that* F

is convex with a differentiable density function. Then the optimal reserve price $\mathcal{R}^*_{m,n,\theta^*} \equiv \mathcal{R}^*$ *decreases in the number of agents* n *and increases in the number of objects* m*.*

The above result yields some empirically testable predictions about the changes in the reserve price as a result of changes in demand (the number of bidders) or in supply (the number of units).[20] This should be compared to the standard "knife-edge" result obtained in the linear case, whereby the reserve price is non-responsive to such changes: since in that case optimal cutoff type and optimal reserve price coincide, this non-response is indeed suggested by our results that display comparative statics in opposite directions for the general model where these two objects do not coincide.

6 Illustrations

In this Section we offer several applications to particular functional forms for values $v(a,\theta)$ and costs $c(a)$. These applications extend and unify previous observations that were obtained in the literature by "ad-hoc" methods.

In addition to the adjustment of reserve price in response to demand and supply, designers may use other practical measures in order to increase revenue. We also illustrate here two such measures: restricting entry ex-ante, and split awards. All derivations for this section are in the Online Appendix.

6.1 Weakly Convex Super-modular Environments

While the regularity of the standard virtual valuation J in the linear case is sufficient for an auction with reserve price to be revenue maximizing, it is not necessary.[21] We can also relax the CSM requirement, which turns out to be useful for the applications discussed below.

Definition 2 (Weak Convex Supermodularity)**.** *We say that the environment is "weakly convex super-modular" (wCSM) if the positive part of the virtual utility,* $\max\{H(p,\theta),0\}$*, is convex in* p *and super-modular in* (p,θ)*.*

In the Online Appendix we prove the following generalization of Theorem 1.

[20] For particular functional forms (see the multiplicative case below) the comparative static result holds even though the above assumptions are not satisfied.

[21] To see this, consider the case where J decreases for some types to which J assigns a negative virtual value. As agents with those types are excluded anyhow, their virtual value plays no role in determining the optimal allocation.

Corollary 3. *Suppose the environment is weakly convex super-modular. Then, the revenue maximizing allocation allocates* m *objects to the agents with the highest types, conditional on these exceeding* $\theta^*_{m,n} \equiv \theta^*$ *where* θ^* *is given by*

$$H(\psi_{m,n}(\theta^*), \theta^*) = 0.$$

Moreover, in the optimal mechanism the expected utility of the lowest type, $\underline{\theta}$ *is zero.*

6.2 Additively Separable Investments

Assume that the costly action $a \in A \subseteq \mathbb{R}_+$ additively increases the agent's value θ for a unit of the good[22]

$$v(a, \theta) = a + \theta.$$

We do not impose any restriction on the set of actions A, and allow it be either finite or infinite. The induced non-linear utility h in the environment where the agent takes an action is of the form

$$h(p, \theta) = \max_{a \in A} [p\,(a + \theta) - c(a)\,].$$

Take an arbitrary optimal selection $a^*(p) \in \arg\max_{a \in A}[p\,a - c(a)]$. Then, h is given by

$$h(p, \theta) = p\,\theta + p\,a^*(p) - c(a^*(p)) = p\,\theta + g(p)\,,$$

where $g(p) := p\,a^*(p) - c(a^*(p))$. As h is convex and increasing in p, the function g is convex. Furthermore, $g(0) = 0$ due to the existence of the costless action $a = 0$. By plugging in (5) we obtain that the virtual utility H is given by

$$H(p, \theta) = p\,J(\theta) + g(p)$$

where $J(\theta) = \theta - \frac{1-F(\theta)}{f(\theta)}$ is the standard virtual value (a la Myerson). As g is convex, it follows immediately that H is convex in p. Furthermore, if J is increasing then H is super-modular, and thus the environment is CSM (Definition 1). Hence, Theorem 1 and Proposition 5 imply the following characterization of the optimal mechanism:

Corollary 4. *Assume that the standard virtual value* $J(\theta) = \theta - \frac{1-F(\theta)}{f(\theta)}$ *is increasing.*

[22] The two-agent, one object, additive case with a quadratic cost is treated in Zhang (2017). She also studies asymmetric mechanisms in that framework (that are not covered here).

1. *Then the environment with additively separable investments is convex super-modular, and both the Uniform-Price and the Pay-Your-Bid auction with reserve price*

$$\mathcal{R}^*_{m,n} = \frac{1 - F(\theta^*_{m,n})}{f(\theta^*_{m,n})}$$

where $\theta^*_{m,n}$ *satisfies*

$$\theta^*_{m,n} + \frac{g(\psi_{m,n}(\theta^*_{m,n}))}{\psi_{m,n}(\theta^*_{m,n})} = \frac{1 - F(\theta^*_{m,n})}{f(\theta^*_{m,n})}$$

are symmetric, revenue maximizing mechanisms.

2. *If the hazard rate* $\frac{f(\theta)}{1-F(\theta)}$ *is increasing, then the optimal reserve price increases in the number of objects and decreases in the number of bidders.*

The second part of the corollary immediately follows from the above characterization of the reserve price together with Proposition 5.

6.3 Multiplicative Separable Investments

In this section we assume that the action $a \in A = \mathbb{R}_+$ increases the agent's value multiplicatively[23]

$$v(a, \theta) = a\,\theta$$

and that the cost function is monomial

$$c(a) = b\frac{a^l}{l} \text{ where } l \geq 2 \text{ and } b > 0.$$

The induced non-linear valuation h is of the form

$$\begin{aligned} h(p, \theta) &= \max_{a \in A} \left(p\,a\,\theta - b\frac{a^l}{l} \right) \\ &= \frac{l-1}{l} \left(\frac{p^l \theta^l}{b} \right)^{\frac{1}{l-1}} . \end{aligned}$$

[23]While formally the set of actions is not compact here this is inconsequential: this assumption was only used to ensure that h is well defined, which we verify explicitly in this application.

It follows from plugging in (5) that the virtual utility H is given here by

$$\begin{aligned} H(p,\theta) &= h(p,\theta) - \frac{\partial h(p,\theta)}{\partial \theta} \times \frac{1-F(\theta)}{f(\theta)} \\ &= p^{\frac{l}{l-1}} \left(\frac{\theta}{b}\right)^{\frac{1}{l-1}} \left(\frac{l-1}{l}\theta - \frac{1-F(\theta)}{f(\theta)}\right). \end{aligned}$$

Assume now that the function

$$K(\theta) := \frac{l-1}{l}\theta - \frac{1-F(\theta)}{f(\theta)} \tag{19}$$

is increasing (a sufficient condition is the standard monotone hazard rate condition).[24] Note that

$$H(p,\theta) = p^{\frac{l}{l-1}} \left(\frac{\theta}{b}\right)^{\frac{1}{l-1}} K(\theta)$$

and thus assuming $\Theta = [0, \overline{\theta}]$ we obtain that $K(0) = -\frac{1}{f(0)} < 0$ and that $H(p,\theta) \geq 0$ implies that $K(\theta) \geq 0$. Hence, $\max\{H(p,\theta), 0\}$ is convex in p and super-modular in (p,θ), i.e. the environment is weakly convex super-modular (see Definition 2). Corollary 3 implies then the following characterization of the optimal mechanism:

Corollary 5. *Assume that the function K defined in (19) is increasing. Then, the environment with multiplicative separable investments and monomial cost is weakly convex super-modular.*

1. *Both the Uniform Price and the Pay-Your-Bid auctions with reserve price*

$$\mathcal{R}^*_{m,n} = \frac{l-1}{l}\left(\frac{\psi_{m,n}(\theta^*)(\theta^*)^l}{b}\right)^{\frac{1}{l-1}}$$

   where θ^* solves $K(\theta^*) = 0$ are symmetric revenue maximizing mechanisms.

2. *Since the critical cutoff θ^* is here independent of both n and m, the optimal reserve price is decreasing in the number of bidders n and increasing in the number of objects m.*

Remark: The same observation and comparative statics results apply to any valuation

[24] For example, assuming an uniform distribution of types, we get $K(\theta) = \frac{2l-1}{l}\theta - 1$.

function that is multiplicatively separable $h(p,\theta) = a(p)b(\theta)$, and that yields a (weakly) convex super-modular environment.

6.4 Fixed Entry Costs[25]

This is also a multiplicative case, but there are only two actions: Enter ($a = 1$) and Stay Out ($a = 0$). Values are given by $v(a_i, \theta_i) = a_i\theta_i$ and costs by $c(a_i) = ca_i$ where $c > 0$.

In contrast to our general model, the action a is here observable. The advantage of an observable action is that the designer can condition the allocation and the transfers on it. Yet, here this additional instrument is not useful since the allocation and transfers are only relevant if this bidder chooses action $a = 1$. Thus, only the mechanisms we already analyzed are applicable.

We obtain

$$h(p,\theta) = \max\{p\theta - c, 0\} \text{ and } \frac{\partial h(p,\theta)}{\partial\theta} = \begin{cases} 0 \text{ if } p\,\theta < c \\ p \text{ if } p\,\theta > c \end{cases} .$$

Thus

$$H(p,\theta) = h(p,\theta) - \frac{\partial h(p,\theta)}{\partial\theta} \times \frac{1 - F(\theta)}{f(\theta)} = \begin{cases} 0 & \text{if } p\,\theta < c \\ p \times \left(\theta - \frac{1-F(\theta)}{f(\theta)}\right) - c & \text{if } \ p\,\theta > c \end{cases} .$$

Assuming an increasing virtual value $J(\theta) = \theta - \frac{1-F(\theta)}{f(\theta)}$, note that $pJ(\theta) - c \geq 0$ implies that $p\theta \geq c$, and thus that

$$\max\{0, H(p,\theta)\} = \max\left\{0, p \times \left(\theta - \frac{1 - F(\theta)}{f(\theta)}\right) - c\right\}$$

is convex in p and super-modular in (p,θ). The environment is weakly convex super-modular.

Corollary 6. *Assume that the standard virtual value $J(\theta) = \theta - \frac{1-F(\theta)}{f(\theta)}$ is increasing. Then, the environment with fixed entry cost is weakly convex super-modular.*

1. *The Uniform price and the Pay-Your-Bid Auctions with reserve price*

$$\mathcal{R}^*_{m,n} = \frac{1 - F(\theta^*_{m,n})}{f(\theta^*_{m,n})}$$

[25] Such auctions are studied by Menezes and Monteiro (2000) and by Celik and Ylankaya (2009) – the latter authors also study asymmetric mechanisms

where $\theta^*_{m,n}$ solves

$$\psi_{m,n}\left(\theta^*_{m,n}\right)\left(\theta^*_{m,n} - \frac{1-F(\theta^*_{m,n})}{f(\theta^*_{m,n})}\right) = c$$

are symmetric revenue maximizing mechanisms.

2. *If the hazard rate $\frac{f(\theta)}{1-F(\theta)}$ is increasing, then the optimal reserve price is decreasing in the number of agents n and increasing in the number of objects m.*

In the Online Appendix we offer another illustration about how our general optimization method can be applied to crowd-sourcing contests.

6.5 Restricting Entry

The symmetric optimal mechanism need not be the overall optimal one. But, in many applications the designer is constrained by the legal system to use nondiscriminatory schemes. Nevertheless, by invoking various technical requirements, minimal standards or financial viability tests, etc.. it is often possible to artificially limit the number of participants in the auction while still treating symmetrically those bidders that do participate. In the standard auction with exogenous valuations limiting participation is never beneficial. But, such a measure may be revenue-improving here: there is an additional positive effect, caused by the upwards adjustment of investments made by those bidders that are selected to participate.[26]

Since our characterization of the optimal mechanism specifies the optimal symmetric mechanism, and derives the expected revenue for **any** number of bidders, it constitutes the main step towards determining the optimal number of bidders in a symmetric mechanism. This number necessarily depends on the particular parameters and functional forms of the values and costs.

Example 1. *Consider the single unit ($m = 1$), additively separable environment with $c(a) = b\frac{a^2}{2}$ for some $b > 0$. Then $g(p) = \frac{p^2}{2b}$ and $H(p, \theta) = J(\theta)p + \frac{p^2}{2b}$. Assuming uniformly distributed types ($\theta \sim U([0, 1])$ the expected revenue is*

$$ER(n) = \frac{n-1}{n+1} - \frac{2n}{n+1}\theta^{*^{n+1}} + \theta^{*^n} + \frac{n}{2b}\frac{1-\theta^{*^{2n-1}}}{2n-1}$$

where $\theta^*_n \equiv \theta^*$ is the solution to

$$2\theta^* - 1 + \frac{(\theta^*)^{n-1}}{2b} = 0.$$

[26]Recall the examples were only two fims were selected to participate

Plotting the revenue of the optimal mechanism shows that, if the investment cost is low enough, the revenue sometimes decreases in the number of agents (c.f. Figure 2). But, the marginal expected revenue

$$\frac{\partial ER(n)}{\partial n} = \frac{2\left(1-{\theta^*}^{n+1}\right)}{(n+1)^2} - \frac{\left(1-{\theta^*}^{2n-1}\right)}{2b\,(2n-1)^2} - \frac{2n{\theta^*}^{n+1}}{n+1}\ln\theta^* + {\theta^*}^{n}\ln\theta^* - \frac{n{\theta^*}^{2n-1}}{(2n-1)\,b}\ln\theta^*$$

is always positive for sufficiently high b.[27] *In that case, the cost of investing becomes prohibitive, and we approach the standard setting without investments where it is well known that revenue is increasing in the number of agents. To see this observe that* $\lim_{b\to\infty}\theta^*_n = 1/2$, *and hence*

$$\lim_{b\to\infty}\frac{\partial ER(n)}{\partial n} = \frac{2\left(1-(0.5)^{n+1}\right)}{(n+1)^2} - \frac{2n\,(0.5)^{n+1}}{n+1}\ln 0.5 + (0.5)^{n}\ln 0.5 > 0.$$

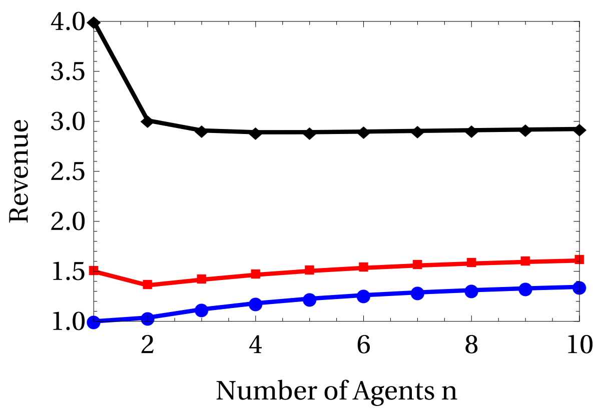

Figure 2: Revenue for the additively separable environments with quadratic costs $c(a) = b\frac{a^2}{2}$ and $\theta \sim U([0,1])$ in the optimal mechanism for $b = 1/2$ (blue), $b = 1/3$ (red), $b = 1/8$ (black).

The above example shows that the celebrated result due to Bulow and Klemperer (1996) need not to hold in our environment. That result states that, in the setting without investments, the seller is always better-off by attracting an additional bidder and setting a zero reserve price rather than setting the optimal reserve price when facing one less bidder. Here the seller may be worse off if an additional bidder participates.

Contrasting the above, additively separable investment case, in the multiplicative case

[27] We look here at the natural extension of R to the real line which allows us to express the derivative with respect to the number of agents.

with a single object, the expected revenue is monotonically increasing in the number of agents under relatively general assumptions.

Lemma 3. *Assume that* $m = 1$*. Assume further that* $v(a,\theta) = a\theta$*,* $a \in A = R$*,* $c(a) = b\frac{a^l}{l}$ *where* $l \geq 2$*,* $b > 0$ *, that* $K(\theta) = \frac{l-1}{l}\theta - \frac{1-F(\theta)}{f(\theta)}$ *is increasing, and that* F *is concave. Then, the expected revenue increases in the number of agents* n*.*

6.6 Split Awards

Another well-known strategy in large procurement exercises is to split the award among two firms, say. On the one hand, having one prize is the optimal structure for fixed values as it allows to extract higher revenues from the highest type; on the other hand, splitting the award increases incentives to invest, as it increases the chances to get an award. When the investment costs are very low, the second effect dominates making split awards beneficial.

Example 2. *Assume additive separable preferences of the form* $v(a,\theta) = a+\theta$ *and quadratic costs* $c(a) = b\frac{a^2}{2}$*. Assume also that types are distributed on the interval* $[0,1]$*. We consider two scenarios:*

1. *A single "large" contract is allocated to an agent who values it at* $2v = 2(a+\theta)$
2. *The contract is split in two, and the two parts are allocated to two agents who value it at* $v = a+\theta$ *each.*

We analyze in the Online Appendix two limit cases that yield contrasting results:

Case 1: b is large enough. Here costs become very high and we approach the standard model without investments. If the virtual value is monotonic, it is well known that a single award generates higher expected revenue: the split award corresponds to a random mechanism that allocates the object with equal probabilities between the two highest bidders, which is suboptimal.

Case 2: b is close enough to zero. Here costs are very low and splitting the contract generates higher expected revenue if $n \geq 3$, while the single contract generates higher expected revenue if $n \leq 3$.[28]

[28] The mechanisms are equivalent for $n = 3$.

7 Ironing

We briefly consider here the case where the super-modularity condition that is necessary in order to apply the Fan-Lorentz theorem, $\frac{\partial H}{\partial\theta\partial p} \geq 0$, does not hold everywhere. Then the monotonicity constraint may bind.[29] In that case, we need to get a deeper insight into the set of extreme points of the set

$$\Omega_{mon}(\phi_{m,n}) = \left\{ p : p \prec_w \phi_{m,n} = \sum_{i=0}^{m-1} \binom{n-1}{i} t^{n-1-i}(1-t)^i \text{ , and } p \text{ non-decreasing} \right\}.$$

This is the set of functions p for which there is no function $u(t) \neq 0$ such that both $p+u$ and $p-u$ are non-decreasing and satisfy

$$p + u \prec_w \phi_{m,n} \text{ and } p - u \prec_w \phi_{m,n}.$$

The set of extreme points of $\Omega_{mon}(\phi_{m,n})$ contains additional elements compared to the set of extreme points of the orbit $\Omega(\phi_{m,n})$ characterized by Ryff (1967). Two general observations can be made:

1. If p is an extreme point of $\Omega_{mon}(\phi_{m,n})$ and p is increasing on a certain interval, then p must coincide with $\phi_{m,n}$ on this interval - in particular, the allocation must be efficient there. This is Lemma 4 below.

2. Assume that $p \in \Omega_{mon}(\phi_{m,n})$ equals a constant κ on an interval $[t_1, t_2]$ while $p(t) < \kappa$ for $t < t_1$ and $p(t) > \kappa$ for $t > t_2$ (recall that we only consider monotonic p). By majorization, we must have $p(t_2) < \phi_{m,n}(t_2)$. If $p < \phi_{m,n}$ on the entire interval $[t_1, t_2]$, then p cannot be an extreme point since the majorization condition is not tight on this interval. Assume then that $p(t_3) = \phi_{m,n}(t_3)$ for an interior point $t_3 \in (t_1, t_2)$.[30] For p to be an extreme point, the majorization constraint must be binding:

$$\begin{aligned} \int_{t_1}^{t_3} (\kappa - \phi_{m,n}(t))d\theta &= \int_{t_3}^{t_2} (\phi_{m,n}(t) - \kappa)d\theta \Leftrightarrow \\ \kappa &= \frac{\int_{t_1}^{t_2} \phi_{m,n}(t)dt}{t_2 - t_1}. \end{aligned}$$

[29] For ironing in the single-agent case with a utility that is **concave** in the allocation, see Toikka (2011).

[30] If p is not continuous the argument needs to be slightly adjusted. A monotonic function is not continuous at most at a countable set of points.

In other words, the "ironed" value κ is completely determined by the interval where p is constant, and by the efficient allocation function $\phi_{m,n}$.[31]

Lemma 4 (Ironing)**.** *Assume that the function p has a finite number of discontinuities on interval $\left[\underline{t}, \overline{t}\right]$. If the function p is an extreme point of $\Omega_{mon}(\phi_{m,n})$, and if it is continuous and strictly increasing on an interval $[t', t'']$, then $p(t) = \phi_{m,n}(t)$ for $t \in [t', t'']$.*

Remark. When comparing different extreme points, the seller chooses between the efficient allocation and a constant probability of allocation over an interval of types: by switching from the efficient allocation to a constant probability, the seller decreases the probability of allocating an unit to the higher types and increases the probability of allocation to the lower types. The super-modularity condition $\frac{\partial^2 H}{\partial p \partial \theta} \geq 0$ implies that the marginal revenue $\frac{\partial H}{\partial p}$ is an increasing function of agent's type: in other words, it is more beneficial to increase the winning probability of the higher than of the lower type. Therefore, the optimal extreme point is the efficient allocation. The inverse intuition implies that, if H is sub-modular $\frac{\partial^2 H}{\partial p \partial \theta} < 0$, the optimal mechanism reduces to an overall constant probability of obtaining the object.

8 Conclusion

We have analyzed revenue maximization in a multi-unit auction framework where agents make, prior to the auction, costly investments that affect their values. The effective agents' utility functions become convex in probability of the physical allocation. Our main results employed a novel maximization approach, focused on a majorization inequality. We identified the revenue maximizing allocation within the class of symmetric mechanisms, and illustrated how it can be implemented via standard auction formats. Finally, we displayed novel comparative statics about the optimal reserve price, and illustrated our results in several specific environments.

References

[2004] Arozamena, L., & Cantillon, E. (2004). Investment Incentives in Procurement Auctions. *The Review of Economic Studies*, **71**(1), 1-18.

[31] Analogous observations (without using majorization) were made in a one-object, discrete type setting by Vohra (2011) and for step allocation functions by Manelli and Vincent (2010).

[1989] Anton, J. J., & Yao, D. A. (1989). Split Awards, Procurement, and Innovation. *The RAND Journal of Economics*, 538-552.

[1997] Bag, P. K. (1997). Optimal Auction Design and R&D. *European Economic Review*, **41**(9), 1655-1674.

[1958] Bauer, H. (1958): "Minimalstellen von Funktionen und Extremalpunkte I" *Archiv der Mathematik* **9**, 389-393.

[2018] Entscheidung der Präsidentenkammer der Bundesnetzagentur, 26.11.2018, über die Festlegungen und Regeln im Einzelnen (Vergaberegeln) und über die Festlegungen und Regelungen für die Durchführung des Verfahrens (Auktionsregeln) zur Vergabe von Frequenzen in den Bereichen 2GH zund 3,6GHz.

[1991] Border, K. (1991): "Implementation of Reduced Form Auctions: A Geometric Approach," *Econometrica* **59**(4), 1175-1187.

[1996] Bulow, J. and Klemperer P. (1996): "Auctions vs Negotiations," *The American Economic Review* **86**(1), 180-194.

[2013] Che, Y.K., Kim, J. and Mierendorff, K. (2013): "Generalized Reduced Form Auctions: A Network-Flow Approach," *Econometrica* **81**(6), 2487–2520.

[2009] Celik, G and Yilankaya, O. (2009): "Optimal Auctions with Simultaneous and Costly Participation", *B.E. Journal of Theoretical Economics (Advances)* **9**(1), Article 24.

[2015] Chawla, S., Hartline, J and Sivan, B. (2015): "Optimal Crowd-sourcing Contests," *Games and Economic Behavior, forthcoming.*

[2003] Che, Y. K., & Gale, I. (2003). Optimal Design of Research Contests. *American Economic Review*, **93**(3), 646-671.

[1990] Crawford, V. (1990): "Equilibrium without Independence," *Journal of Economic Theory* **50**, 127-154.

[2017] Dillenberger, D. and Raymond, C. (2017) :"Mixture Aversion and the Consensus Effect,", discussion paper, University of Pennsylvania.

[2014] *Directive 2014/24/EU of the European Parliament and of the Council of 26 February 2014 on public procurement and repealing Directive 2004/18/EC*

[2009] Fullerton, R. L., & McAfee, R. P. (1999). Auctioning Entry Into Tournaments. *Journal of Political Economy*, **107**(3), 573-605.

[1957] Gale, D. (1957) "A Theorem on Flows in Networks," *Pacific Journal of Mathematics*, **7**, 1073-1082.

[2013] Gershkov, A., Goeree J., Kushnir, A., Moldovanu, B. and Shi, X. (2013): "On the Equivalence of Bayesian and Dominant Strategy Implementation," *Econometrica* **81**(1), 197-220.

[2012] Gong, J., Li, J., and McAfee, R.P. (2012): "Split-Award Contracts with Investment," *Journal of Public Economics,* **96**, 188-197.

[1984] Guesnerie, R. and Laffont J.-J. (1984): "A Complete Solution to a Class of Principal-Agent Problems with an Application to the Control of a Self-Managed Firm," *Journal of Public Economics* **25**, 329-369.

[1954] Fan, K. and Lorentz, G.G. (1954): "An Integral Inequality," *The American Mathematical Monthly* **61**(9), 626-631

[1991] Fudenberg, D. and Tirole, J. (1991): *Game Theory*, Cambridge: MIT Press.

[1929] Hardy, G. H., Littlewood J. E., and Polya G. (1929): "Some Simple Inequalities Satisfied by Convex Functions," *Messenger Math* **58**, 145-152.

[2015] Hart, S. and Reny, P (2015): "Implementation of Reduced Form Mechanisms: a Simple Approach and a New Characterization," *Economic Theory Bulletin* **3**(1), 1-8.

[2018] Hatfield, J., Kojima, F and Kominers, S. (2018): "Strategy-Proofness, Investment Efficiency, and Marginal Returns: An Equivalence," *working paper.*

[1979] Kahneman, D. and Tversky, A. (1979): "Prospect Theory: An Analysis of Decision Under Risk," *Econometrica* **47**(2), 263–291.

[1989] Karni, E. and Safra, Z. (1986): "Dynamic Consistency, Revelations in Auctions and the Structure of Preferences," *Review of Economic Studies* ***56***,421-434.

[1992] King, I., Welling, L., & McAfee, R. P. (1992). Investment Decisions Under First and Second Price Auctions. *Economics Letters*, **39**(3), 289-293.

[1979] Kreps, D. and Porteus, E. (1979): "Temporal von Neumann-Morgenstern and Induced Preferences," *Journal of Economic Theory* **20**, 8l-109.

[2006] Li, Y., Lovejoy, W., & Gupta, S. (2006). Pre-auction Investments by Type-conscious Agents, Ross School of Business, WP 1057, U. of Michigan

[1949] Lorentz, G.G. (1949): "A Problem of Plane Measure," *American Journal of Mathematics* **71**, 417-426.

[1984] Machina, M. (1984): "Temporal Risk and the Nature of Induced Preferences," *Journal of Economic Theory* **33**, 199-231.

[2010] Manelli, A. and Vincent, D. (2010): "Bayesian and Dominant Strategy Implementation in the Independent, Private Values Model," *Econometrica* **78**(6)*,* 1905-1939.

[1984] Maskin, E. S. and Riley, J. (1984), "Optimal Auctions with Risk-Averse Buyers," *Econometrica* **52**, 1473–1518.

[1984] Matthews, S. A. (1984), "On the Implementability of Reduced Form Auctions," *Econometrica* **52**, 1519–1522.

[2000] Menezes, F. and Monteiro, P. (2000): "Auctions with Endogenous participation," *Review of Economic Design* **5**, 71-89.

[2017] *Discovering Prices*, New York: Columbia University Press

[1994] Milgrom, P. and Shannon C. (1994) "Monotone Comparative Statics," *Econometrica*, 157–180.

[1981] Myerson, R.B. (1981) "Optimal Auction Design," *Mathematics of Operations Research*, **6**(1), 58–73.

[1984] Neilson, W. (1994) "Second Price Auctions without Expected Utility," *Journal of Economic Theory* **62**, 136-151

[1996] Piccione, M., & Tan, G. (1996). Cost-reducing Investment, Optimal Procurement and Implementation by Auctions. *International Economic Review*, 663-685.

[1982] Quiggin, J. (1982), "A Theory of Anticipated Utility," *Journal of Economic Behavior & Organization*, **3**(4), 323-343.

[1981] Riley, J. and Samuelson, W. (1981), "Optimal Auctions," American Economic Review, **71**(3), 381-392.

[1967] Ryff, J.V. (1967): " Extreme Points of Some Convex Subsets of $L_1(0, 1)$," *Proceedings of the American Mathematical Society* **18**(6), 1026-1034.

[1957] Ryser, H. J. (1957) "Combinatorial Properties of Matrices of Zeros and Ones," *Canadian Journal of Mathematics* **9**, 371-377.

[1965] Strassen, V. (1965): "The Existence of Probability Measures with Given Marginals," *Mathematical Statistics* **36**, 423-439.

[2011] Vohra, R.V. (2011): *Mechanism Design: A Linear Programming Approach*, Cambridge: Cambridge University Press

[1992] Tan, G. (1992). Entry and R & D in Procurement Contracting. *Journal of Economic Theory*, **58**(1), 41-60.

[2011] Toikka, J. (2011) : "Ironing Without Control," *Journal of Economic Theory*, **146**(6), 2510–2526.

[2017] Zhang, M. (2017): "Auctions vs. Negotiations: Optimal Selling Mechanism with Endogenous Bidder Values," discussion paper, University of Technology, Sydney.

Appendix Proofs

Proof of Lemma 1: h is convex in p as it is the maximum over functions that are linear in p. It is increasing in p and θ as $p\, v(a, \theta) - c(a)$ is increasing in p and θ. By assumption there exists an action 0 with $c(0) = 0$ and the cost is always non-negative. Hence, $h(0, \theta) = \max_{a \in A_i} -c(a) = 0$. Finally, let $a^*(p_i, \theta_i)$ be an optimal selection for the problem given in (3). As $p_i\, v(a_i, \theta_i) - c(a_i)$ is super-modular in (a_i, θ_i) we can always pick an optimal selection $a^*(p_i, \theta_i)$ which is non-decreasing in θ_i (Theorem 4 in Milgrom and Shannon (1994)). By the Envelope Theorem, we obtain that h is absolutely continuous in p with (weak) derivative

$$\frac{\partial}{\partial p_i} h(p_i, \theta_i) = v(a^*(p_i, \theta_i), \theta_i)\,. \tag{20}$$

As $a^*(p_i, \theta_i)$ is non-decreasing in θ_i and v is non-decreasing in a_i and θ_i, the marginal utility associated with an increase in the probability of receiving the object, $\frac{\partial}{\partial p_i} h$, is non-decreasing in θ_i. Thus, h is super-modular in (p_i, θ_i).

Proof of Proposition 1: Denote by $p(\theta_i)$ the interim probability with which agent i receives an object when she is of type θ_i and by

$$u(\theta_i, \tilde{\theta}_i) = h(p(\tilde{\theta}_i), \theta_i) - \mathbb{E}\left[y(\tilde{\theta}_i, \theta_{-i})\right]$$

the interim utility of agent i if she is of type θ_i but she misrepresents her type as $\tilde{\theta}_i$. Recall that we only consider symmetric mechanisms, and thus the functions p and u are independent of the agents' identities- we drop the agent's sub-index i for the remainder of the proof. Let $U(\theta)$ be the indirect utility of an agent of type θ who reports optimally

$$U(\theta) = \max_{\tilde{\theta}} u(\theta, \tilde{\theta}).$$

As it is optimal to report truthfully in a direct, incentive compatible mechanism, the envelope theorem yields that

$$\frac{\partial U}{\partial \theta} = \frac{\partial u}{\partial \theta} = \frac{\partial h}{\partial \theta}.$$

The existence of $\frac{\partial h}{\partial \theta}$ follows by the assumed Lipschitz continuity of v in θ. This implies that we can represent an agent's interim utility as

$$U(\theta) = U(\underline{\theta}) + \int_{\underline{\theta}}^{\theta} \frac{\partial h(p(z), z)}{\partial \theta} dz = \int_{\underline{\theta}}^{\theta} \frac{\partial h(p(z), z)}{\partial \theta} dz$$

where the last equality follows from the binding participation constraint.

The revenue from each type is then given by

$$h(p(\theta), \theta) - U(\theta) = h(p(\theta), \theta) - \int_{\underline{\theta}}^{\theta} \frac{\partial h(p(z), z)}{\partial \theta} dz$$

and thus the objective becomes

$$\max_{p} \int_{\underline{\theta}}^{\overline{\theta}} \left(h(p(\theta), \theta) - \int_{\underline{\theta}}^{\theta} \frac{\partial h(p(z), z)}{\partial \theta} dz \right) f(\theta) d\theta$$

Using integration by parts, we obtain:

$$\int_{\underline{\theta}}^{\overline{\theta}} \left(\int_{\underline{\theta}}^{\theta} \frac{\partial h(p(z), z)}{\partial \theta} dz \right) f(\theta) d\theta = \int_{\underline{\theta}}^{\overline{\theta}} \frac{\partial h(p(\theta), \theta)}{\partial \theta} [1 - F(\theta)] d\theta$$

so that the expected revenue from any agent is only a function of the interim probability p with which an agent receives an object

$$\int_{\underline{\theta}}^{\overline{\theta}} \left(h(p(\theta), \theta) - \frac{\partial h(p(\theta), \theta)}{\partial \theta} \times \frac{1 - F(\theta)}{f(\theta)} \right) f(\theta) d\theta \, .$$

Multiplying by the number of agents n yields the result.

Proof of Lemma 2:

$$\begin{aligned}
\frac{\partial}{\partial t} \left(\sum_{i=0}^{n} \min\{i, m\} \binom{n}{i} t^{n-i} (1-t)^i \right) &= \frac{\partial}{\partial t} \left(m - \sum_{i=0}^{m-1} (m-i) \binom{n}{i} t^{n-i} (1-t)^i \right) \\
&= - \sum_{i=0}^{m-1} (m-i) \binom{n}{i} \left[(n-i) t^{n-i-1} (1-t)^i - i t^{n-i} (1-t)^{i-1} \right] \\
&= \sum_{i=0}^{m-1} \left[(m-i) \binom{n}{i} i t^{n-i} (1-t)^{i-1} \right] - \left[(m-i) \binom{n}{i} (n-i) t^{n-i-1} (1-t)^i \right] \\
&= \sum_{i=0}^{m-1} \left[(m-i-1) \binom{n}{i+1} (i+1) t^{n-i-1} (1-t)^i \right] - \left[(m-i) \binom{n}{i} (n-i) t^{n-i-1} (1-t)^i \right] \\
&= \sum_{i=0}^{m-1} t^{n-i-1} (1-t)^i \left[(m-i-1) \binom{n}{i+1} (i+1) - (m-i) \binom{n}{i} (n-i) \right] \\
&= \sum_{i=0}^{m-1} t^{n-i-1} (1-t)^i \left[(m-i-1) \binom{n}{i} (n-i) - (m-i) \binom{n}{i} (n-i) \right] \\
&= - \sum_{i=0}^{m-1} t^{n-i-1} (1-t)^i \binom{n}{i} [n-i] = -n \sum_{i=0}^{m-1} t^{n-i-1} (1-t)^i \binom{n-1}{i} = -n \, \phi(t) \, .
\end{aligned}$$

Finally, we obtain that $\int_0^1 \phi(t) dt$ is given by

$$n \int_0^1 \phi(t) dt = \sum_{i=0}^{n} \min\{i, m\} \binom{n}{i} t^{n-i} (1-t)^i \Big|_{t=0} = m \, . \quad \square$$

Lemma 5. *$G(0, t) = 0$ for all $t \in [0, 1]$. Furthermore, if the environment is CSM, then*

$G(q,t)$, expressed as a function of the quantile, is convex in q and super-modular in (q,t).

Proof of Lemma 5: Note first that

$$G(0,t) = H(0, F^{-1}(t)) = h(0, F^{-1}(t)) - \frac{\partial h(0, F^{-1}(t))}{\partial \theta} \times \frac{1-t}{f(F^{-1}(t))}.$$

As $h(0,\theta) = 0$ by assumption, it follows that $\frac{\partial h(0,F^{-1}(t))}{\partial \theta} = 0$ and thus that $G(0,t) = 0$. In a CSM environment $G(q,t) = H(q, F^{-1}(t))$ is convex in q since $H(p,\theta)$ is convex in p. Similarly, because H is super-modular, because F^{-1} is strictly increasing and because super-modularity is preserved under strictly monotone transformations, we obtain that G is super-modular.

Proof of Theorem 1: Since the number of agents and objects is fixed throughout the proof and thus there is no risk of confusion, we will drop the subindices, and write ϕ for $\phi_{m,n}$ throughout. Let $q^*(t)$ be the interim probability with which an agent of type $\theta = F^{-1}(t)$ receives an object in an optimal symmetric mechanism. Since in a symmetric mechanism no agent can receive the object with an ex-ante probability greater than m/n, we obtain that

$$\int_0^1 q^*(t)dt \leq \frac{m}{n} = \int_0^1 \phi(t)dt\,.$$

Consequently, there exists a quantile $t^* \in [0,1]$ such that the ex-ante probability with which an agent receives an object in the optimal mechanism equals the probability with which an agent with a type higher than $F^{-1}(t^*)$ would receive an object under the interim allocation probability ϕ :

$$\int_0^1 q^*(t)dt = \int_{t^*}^1 \phi(t)dt.$$

Let $\theta^* = F^{-1}(t^*)$ be the type corresponding to the quantile t^* and let $q(t)$ be the interim allocation probability corresponding to the allocation rule that assigns a unit to an agent if and only if her type is above θ^* and at most $m-1$ other agents have a higher valuation, i.e.

$$q(t) = \mathbf{1}_{\{t \geq t^*\}}\phi(t)\,.$$

We now argue that q majorizes q^*. By Condition (13) of Proposition 3 we have that for all $t \in [0,1]$

$$\int_t^1 q^*(z)\mathrm{d}z \leq \int_t^1 \phi(z)\mathrm{d}z\,. \tag{21}$$

By the definition of the critical quantile t^* we obtain that

$$\int_t^1 q^*(z)\mathrm{d}z \le \int_0^1 q^*(z)\mathrm{d}z = \int_{t^*}^1 \phi(z)\mathrm{d}z\,. \tag{22}$$

Combining, (21) and (22) that for all $t \in [0,1]$

$$\int_t^1 q^*(z)\mathrm{d}z \le \min\left\{\int_t^1 \phi(z)\mathrm{d}z, \int_{t^*}^1 \phi(z)\mathrm{d}z\right\} = \int_t^1 \mathbf{1}_{t\ge t^*}\phi(z)\mathrm{d}z = \int_t^1 q(z)\mathrm{d}z\,.$$

As $\int_0^1 q(z)dz = \int_0^1 q^*(z)dz$ by the definitions of t^* and q , we obtain that q majorizes q^*. Note that, as the environment is CSM, Lemma 5 implies that G satisfies the conditions of Proposition 4. By the Fan-Lorentz Theorem (Proposition 4) we thus have that

$$\int_0^1 G(q^*(t),t)dt \le \int_0^1 G(q(t),t)dt\,.$$

But, as q^* is a revenue maximizing interim probability, the above equation must hold with equality. Consequently, every mechanism which implements the interim probability $q(t) = \mathbf{1}_{\{t\ge t^*\}}\phi(t)$ is revenue maximizing.

Finally, observe that, by Lemma 5, $G(0,\cdot)=0$. Thus, the expected revenue as a function of t^* is given by

$$t^* \mapsto \int_0^1 G\left(\mathbf{1}_{\{t\ge t^*\}}\phi(t),t\right)dt = \int_{t^*}^1 G\left(\phi(t),t\right)dt\,. \tag{23}$$

We show now that $\theta \mapsto H(\psi(\theta),\theta)$ changes sign at most once, from negative to positive. Assume that it crosses zero at $\theta^* \neq \underline{\theta}$, that is $H(\psi(\theta^*),\theta^*)=0$. Then for any $\theta > \theta^*$ we have

$$H(\psi(\theta),\theta) - H(\psi(\theta^*),\theta^*) = H(\psi(\theta),\theta) - H(\psi(\theta),\theta^*) + H(\psi(\theta),\theta^*) - H(\psi(\theta^*),\theta^*) \ge 0$$

where the last inequality follows from: (a) the convexity of H which implies that

$$\frac{H(\psi(\theta),\theta^*)}{\psi(\theta)} \ge \frac{H(\psi(\theta^*),\theta^*)}{\psi(\theta^*)} = 0$$

and hence that

$$H(\psi(\theta),\theta^*) - H(\psi(\theta^*),\theta^*) \ge 0\,.$$

and (b) the super-modularity of H which implies that

$$\begin{aligned} H(\psi(\theta),\theta)-H(0,\theta) &\geq H(\psi(\theta),\theta^*)-H(0,\theta^*) \Longleftrightarrow \\ H(\psi(\theta),\theta)-H(\psi(\theta),\theta^*) &\geq H(0,\theta)-H(0,\theta^*)=0. \end{aligned}$$

Since F is a monotone transformation and since $\theta \mapsto H(\psi(\theta),\theta)$ changes sign at most once, from negative to positive, so does $t \mapsto G(\phi(t),t)$. Since $t \mapsto G(\phi(t),t)$ changes its sign at most once, from negative to positive, (23) this function is quasi-concave, and thus the optimal quantile t^* satisfies the first order condition $G(\phi(t^*),t^*)=0$. The result follows because

$$G(\phi(t^*),t^*)=H(\phi(t^*),F^{-1}(t^*))=H((\phi\circ F)(\theta^*),\theta^*)=H(\psi_{m,n}(\theta^*),\theta^*).$$

Proof of Proposition 5: Let $\psi,\psi':(0,1)\rightarrow(0,\infty)$ be continuous functions with $\psi(q)\leq\psi'(q)$ for all $q\in[0,1]$. Recall that $\theta\mapsto H(\psi(\theta),\theta)$, and $\theta\mapsto H(\psi'(\theta),\theta)$ change sign only once[32], from negative to positive, and define θ^*,θ^{**} implicitly by

$$0=H(\psi(\theta^*),\theta^*) \text{ and } 0=H(\psi'\left(\theta^{**}\right)),\theta^{**}).$$

We have that

$$0=H(\psi(\theta^*),\theta^*)=\frac{H(\psi(\theta^*),\theta^*)}{\psi(\theta^*)}\leq\frac{H(\psi'(\theta^*),\theta^*)}{\psi'(\theta^*)},$$

where the last inequality follows from the convexity of H , and from the fact that $H(0,\cdot)=0$. The last inequality implies that $0\leq H(\psi'(\theta^*),\theta^*)$ and, since $\theta\mapsto H(\psi'(\theta),\theta)$ changes sign only once, it follows that $\theta^*\geq\theta^{**}$. As $\phi_{m,n}$ decreases in n and increases in m point-wise, the result follows by setting $\psi=\phi_{m,n}\circ F$.

[32] If $H\left(\psi\left(\theta\right),\theta\right)$ and $H\left(\psi'\left(\theta\right),\theta\right)$ do not change sign, define θ^* and θ^{**} to be the relevant boundary of the interval $\left[\underline{\theta},\overline{\theta}\right]$.

Online Appendix: Not for Publication

Proof of Corollary 3: By Proposition 1 the revenue in any symmetric incentive compatible mechanism is given by

$$n \int_{\Theta} H(p(\theta), \theta)\, f(\theta)\, d\theta\,.$$

Let $\tilde{H}(p, \theta) := \max\{H(p, \theta), 0\}$. Trivially, an upper bound on the optimal revenue is thus given by

$$\max_{p} n \int_{\Theta} \max\{H(p(\theta), \theta), 0\}\, f(\theta)\, d\theta = \max_{p} n \int_{\Theta} \tilde{H}(p(\theta), \theta)\, f(\theta)\, d\theta\,. \tag{24}$$

As this environment is CSM it follows from Theorem 1 that the revenue maximizing mechanism allocates according $\psi = \phi \circ F$, conditional on being above a critical type θ^* that is the unique solution of $H((\phi \circ F)(\theta^*), \theta^*) = 0$. As $H(0, \theta) = 0$ for all θ by Lemma 5 , we obtain that this achieves the upper bound given in (24) and thus is an optimal mechanism.

Proof of Proposition 6: Agent i's utility in a symmetric equilibrium where all bidddrs but i use strategy β and where i submits a bid b is given by

$$h\left(\psi_{m,n}(\beta^{-1}(b)), \theta_i\right) - \psi_{m,n}(\beta^{-1}(b))b\,. \tag{25}$$

Taking the derivative with respect to b yields that

$$\left[\frac{\partial h\left(\psi_{m,n}(\beta^{-1}(b)), \theta_i\right)}{\partial p} - b\right] \frac{\partial \psi_{m,n}(\beta^{-1}(b))}{\partial \theta} \frac{\partial \beta^{-1}(b)}{\partial b} - \psi_{m,n}(\beta^{-1}(b)).$$

Plugging $b = \beta(\theta)$ (as it is optimal for agent i to make the equilibrium bid) yields

$$\left[\frac{\partial h\left(\psi_{m,n}(\theta_i), \theta_i\right)}{\partial p} - \beta(\theta_i)\right] \frac{\partial \psi_{m,n}(\theta_i)}{\partial \theta} \frac{1}{\beta'(\theta_i)} - \psi_{m,n}(\theta_i) = 0\,.$$

Rearranging for β' yields the following differential equation:

$$\beta'(\theta_i) = \frac{\partial \psi_{m,n}(\theta_i)}{\partial \theta} \frac{1}{\psi_{m,n}(\theta_i)} \left[\frac{\partial h\left(\psi_{m,n}(\theta_i), \theta_i\right)}{\partial p} - \beta(\theta_i)\right]. \tag{26}$$

Since $\theta_i \mapsto \frac{\partial \psi_{m,n}(\theta_i)}{\partial \theta} \frac{1}{\psi_{m,n}(\theta_i)}$ is continuous and since the left-hand-side of (26) is uniformly Lipschitz in β, the O.D.E (26) admits a unique solution for every initial value. Set $\theta' \equiv$

$\theta'_{m,n,\mathcal{R}}$ to be the solution to the equation

$$\frac{h\left(\psi_{m,n}(\theta'),\theta'\right)}{\psi_{m,n}(\theta')}=\mathcal{R}\,.$$

If we set $\beta(\theta')=\mathcal{R}$ then the agent with type θ' is, by construction, indifferent between bidding $\mathcal{R}$ and bidding zero. For all higher types $\theta\in(\theta',\overline{\theta}]$ we extend the bidding function by solving the ODE (26). Since (26) is linear, we obtain the explicit solution

$$\beta(\theta_i)=\mathcal{R}\frac{\psi_{m,n}(\theta')}{\psi_{m,n}(\theta_i)}+\frac{1}{\psi_{m,n}(\theta_i)}\int_{\theta'}^{\theta_i}\frac{\partial\psi_{m,n}(z)}{\partial\theta}\frac{\partial h\left(\psi_{m,n}(z),z\right)}{\partial p}dz$$

To verify, take the derivative to obtain

$$\begin{aligned}\beta'(\theta_i)=&-\frac{\partial\psi_{m,n}(\theta_i)}{\partial\theta}\frac{\psi_{m,n}(\theta')}{\left(\psi_{m,n}(\theta_i)\right)^2}\mathcal{R}-\frac{\partial\psi_{m,n}(\theta_i)}{\partial\theta}\frac{1}{\left(\psi_{m,n}(\theta_i)\right)^2}\int_{\theta'}^{\theta_i}\frac{\partial\psi_{m,n}(z)}{\partial\theta}\frac{\partial h\left(\psi_{m,n}(z),z\right)}{\partial p}dz\\&+\frac{\partial\psi_{m,n}(\theta_i)}{\partial\theta}\frac{1}{\psi_{m,n}(\theta_i)}\frac{\partial h\left(\psi_{m,n}(\theta_i),\theta_i\right)}{\partial p}\\&=-\frac{\partial\psi_{m,n}(\theta_i)}{\partial\theta}\frac{\beta(\theta_i)}{\psi_{m,n}(\theta_i)}+\frac{\partial\psi_{m,n}(\theta_i)}{\partial\theta}\frac{1}{\psi_{m,n}(\theta_i)}\frac{\partial h\left(\psi_{m,n}(\theta_i),\theta_i\right)}{\partial p}\\&=\frac{\partial\psi_{m,n}(\theta_i)}{\partial\theta}\frac{1}{\psi_{m,n}(\theta_i)}\left[\frac{\partial h\left(\psi_{m,n}(\theta_i),\theta_i\right)}{\partial p}-\beta(\theta_i)\right].\end{aligned}$$

We first need to show that the solution β of (26) is increasing for all $\theta_i\geq\theta'$. Observe that:

$$\beta'\geq 0\Leftrightarrow\mathcal{R}\frac{\partial\psi_{m,n}(\theta')}{\partial\theta}\frac{1}{\psi_{m,n}(\theta_i)}+\frac{1}{\psi_{m,n}(\theta_i)}\int_{\theta'}^{\theta_i}\frac{\partial\psi_{m,n}(z)}{\partial\theta}\frac{\partial h\left(\psi_{m,n}(z),z\right)}{\partial p}dz\leq\frac{\partial h\left(\psi_{m,n}(\theta_i),\theta_i\right)}{\partial p}$$

Plugging the expression for $\mathcal{R}$, this becomes:

$$\frac{h\left(\psi_{m,n}(\theta'),\theta'\right)}{\psi_{m,n}(\theta_i)}+\frac{1}{\psi_{m,n}(\theta_i)}\int_{\theta'}^{\theta_i}\frac{\partial\psi_{m,n}(z)}{\partial\theta}\frac{\partial h\left(\psi_{m,n}(z),z\right)}{\partial p}dz\leq\frac{\partial h\left(\psi_{m,n}(\theta_i),\theta_i\right)}{\partial p}$$

By super-modularity, we can bound the left hand side above:

$$\begin{aligned}
& \frac{h\left(\psi_{m,n}(\theta'),\theta'\right)}{\psi_{m,n}(\theta_i)} + \frac{1}{\psi_{m,n}(\theta_i)}\int_{\theta'}^{\theta_i} \frac{\partial\psi_{m,n}(z)}{\partial\theta}\frac{\partial h\left(\psi_{m,n}(z),z\right)}{\partial p}dz \\
\leq \quad & \frac{h\left(\psi_{m,n}(\theta'),\theta'\right)}{\psi_{m,n}(\theta_i)} + \frac{\frac{\partial h(\psi_{m,n}(\theta_i),\theta_i)}{\partial p}}{\psi_{m,n}(\theta_i)}(\psi_{m,n}(\theta_i)-\psi_{m,n}(\theta') \\
= \quad & \frac{h\left(\psi_{m,n}(\theta'),\theta'\right)}{\psi_{m,n}(\theta_i)} + \frac{\partial h\left(\psi_{m,n}(\theta_i),\theta_i\right)}{\partial p} - \frac{\psi_{m,n}(\theta')}{\psi_{m,n}(\theta_i)}\frac{\partial h\left(\psi_{m,n}(\theta_i),\theta_i\right)}{\partial p}
\end{aligned}$$

Thus, we need to show that:

$$\begin{aligned}
\frac{h\left(\psi_{m,n}(\theta'),\theta'\right)}{\psi_{m,n}(\theta_i)} + \frac{\partial h\left(\psi_{m,n}(\theta_i),\theta_i\right)}{\partial p} - \frac{\psi_{m,n}(\theta')}{\psi_{m,n}(\theta)}\frac{\partial h\left(\psi_{m,n}(\theta_i),\theta_i\right)}{\partial p} \quad &\leq \quad \frac{\partial h\left(\psi_{m,n}(\theta_i),\theta_i\right)}{\partial p} \Longleftrightarrow \\
\frac{1}{\psi_{m,n}(\theta_i)}[h\left(\psi_{m,n}(\theta'),\theta'\right) - \psi_{m,n}(\theta')\frac{\partial h\left(\psi_{m,n}(\theta_i),\theta_i\right)}{\partial p}] \quad &\leq \quad 0 \Longleftrightarrow \\
\frac{h\left(\psi_{m,n}(\theta'),\theta'\right)}{\psi_{m,n}(\theta')} \quad &\leq \quad \frac{\partial h\left(\psi_{m,n}(\theta_i),\theta_i\right)}{\partial p}
\end{aligned}$$

The last inequality holds by the convexity and super-modularity of h.

We need to verify that it is indeed optimal for the type θ_i to bid $\beta(\theta_i)$. We start by considering types $\theta_i \in [\theta', \overline{\theta}]$. By construction, the bid $\beta(\theta_i)$ satisfies the agent's first order condition. If the agent deviates by making the equilibrium bid that is optimal for type $\hat{\theta}$, her utility equals

$$h\left(\psi_{m,n}(\hat{\theta}),\theta_i\right) - \psi_{m,n}(\hat{\theta})\beta(\hat{\theta})\,. \tag{27}$$

The derivative with respect to $\hat{\theta}$ is

$$\begin{aligned}
&\left[\frac{\partial h\left(\psi_{m,n}(\hat{\theta}),\theta_i\right)}{\partial p} - \beta(\hat{\theta})\right]\frac{\partial\psi_{m,n}(\hat{\theta})}{\partial\theta} - \psi_{m,n}(\hat{\theta})\beta'(\hat{\theta}) \\
&= \underbrace{\left[\frac{\partial h\left(\psi_{m,n}(\hat{\theta}),\hat{\theta}\right)}{\partial p} - \beta(\hat{\theta})\right]\frac{\partial\psi_{m,n}(\hat{\theta})}{\partial\theta}}_{\psi_{m,n}(\hat{\theta})\beta'(\hat{\theta})} - \psi_{m,n}(\hat{\theta})\beta'(\hat{\theta}) + \left[\frac{\partial h\left(\psi_{m,n}(\hat{\theta}),\theta_i\right)}{\partial p} - \frac{\partial h\left(\psi_{m,n}(\hat{\theta}),\hat{\theta}\right)}{\partial p}\right] \\
&= \frac{\partial h\left(\psi_{m,n}(\hat{\theta}),\hat{\theta}\right)}{\partial p} - \frac{\partial h\left(\psi_{m,n}(\hat{\theta}),\hat{\theta}\right)}{\partial p}\,.
\end{aligned} \tag{28}$$

(at the last step we used the fact that β solves the ODE (26). As h is super-modular, expression (28) is increasing, and changes its sign from positive to negative at $\theta_i = \hat{\theta}$. Thus, the agent's objective (27) is concave, and is maximized at $\theta_i = \hat{\theta}$. An agent of type $\theta_i \in [\theta', \overline{\theta}]$ thus prefers to make the bid $\beta(\theta_i)$ over any other bid in $[\beta(\theta'), \beta(\bar{(\theta)}] = [R, \beta(\overline{\theta})]$. Clearly it can never be optimal for the agent to make a bid higher than $\beta(\overline{\theta})$ as a bid of $\beta(\overline{\theta})$ would already ensure that she wins and pays strictly less. It remains to verify that the agent does not want to make a bid of zero: observe that the agent could deviate to bid $\mathcal{R}$, which would yield a utility higher than the equilibrium utility of the type θ'. It thus suffices to verify that the equilibrium utility of the type θ' is non-negative. This type's equilibrium utility is given by

$$h\left(\psi_{m,n}(\theta'), \theta'\right) - \psi_{m,n}(\theta')\mathcal{R}\,,$$

that, by the definition of the reserve price, equals zero. Finally, we verify that no type $\theta_i \in [\underline{\theta}, \theta']$ wants to deviate by making a non-zero bid. To see this, note that by deviating to any bid in $[\mathcal{R}, \beta(\overline{\theta})]$ such a type would get a utility which is lower than the utility the type θ' gets from making this bid. But, by construction, the optimal bid of type $\theta^{'}$ equals $\mathcal{R}$ and yields her a utility of zero. This implies that the utility resulting from any bid greater zero must be less than zero for all lower types. □

Proof of Proposition 7: We treat here m and n as real variables and prove the desired monotonicity properties. It is then clear that they hold also on the domain of the natural numbers.

Recall that the critical cutoff is given by

$$H\left(\psi_{m,n}\left(\theta^*\right), \theta^*\right) = 0.$$

Therefore,

$$\frac{\partial \theta^*}{\partial n} = -\frac{\frac{\partial H}{\partial p}\left(\psi_{m,n}\left(\theta^*\right), \theta^*\right)\frac{\partial \psi_{m,n}}{\partial n}\left(\theta^*\right)}{\frac{\partial H}{\partial \theta}\left(\psi_{m,n}\left(\theta^*\right), \theta^*\right) + \frac{\partial H}{\partial p}\left(\psi_{m,n}\left(\theta^*\right), \theta^*\right)\frac{\partial \psi_{m,n}}{\partial \theta}\left(\theta^*\right)}$$

Also, we know that

$$\mathcal{R}^* = \frac{h\left(\psi_{m,n}\left(\theta^*\right), \theta^*\right)}{\psi_{m,n}\left(\theta^*\right)}.$$

which implies

$$\frac{\partial \mathcal{R}^*}{\partial n} = \frac{\left[\frac{\partial h(\psi_{m,n}(\theta^*),\theta^*)}{\partial p}\left(\frac{\partial \psi_{m,n}(\theta^*)}{\partial n} + \frac{\partial \psi_{m,n}(\theta^*)}{\partial \theta}\frac{\partial \theta^*}{\partial n}\right) + \frac{\partial h(\psi_{m,n}(\theta^*),\theta^*)}{\partial \theta}\frac{\partial \theta^*}{\partial n}\right]\psi_{m,n}(\theta^*)}{\left(\psi_{m,n}(\theta^*)\right)^2} - \frac{\left(\frac{\partial \psi_{m,n}(\theta^*)}{\partial n} + \frac{\partial \psi_{m,n}(\theta^*)}{\partial \theta}\frac{\partial \theta^*}{\partial n}\right) h\left(\psi_{m,n}(\theta^*),\theta^*\right)}{\left(\psi_{m,n}(\theta^*)\right)^2}.$$

The sign of the last derivative is just the sign of

$$\left[\frac{\partial h\left(\psi_{m,n}(\theta^*),\theta^*\right)}{\partial p}\left(\frac{\partial \psi_{m,n}(\theta^*)}{\partial n} + \frac{\partial \psi_{m,n}(\theta^*)}{\partial \theta}\frac{\partial \theta^*}{\partial n}\right) + \frac{\partial h\left(\psi_{m,n}(\theta^*),\theta^*\right)}{\partial \theta}\frac{\partial \theta^*}{\partial n}\right]\psi_{m,n}(\theta^*) - \left(\frac{\partial \psi_{m,n}(\theta^*)}{\partial n} + \frac{\partial \psi_{m,n}(\theta^*)}{\partial \theta}\frac{\partial \theta^*}{\partial n}\right) h\left(\psi_{m,n}(\theta^*),\theta^*\right)$$

$$\begin{aligned} & \left[\frac{\partial h\left(\psi_{m,n}(\theta^*),\theta^*\right)}{\partial p}\left(\frac{\partial \psi_{m,n}(\theta^*)}{\partial n} + \frac{\partial \psi_{m,n}(\theta^*)}{\partial \theta}\frac{\partial \theta^*}{\partial n}\right) + \frac{\partial h\left(\psi_{m,n}(\theta^*),\theta^*\right)}{\partial \theta}\frac{\partial \theta^*}{\partial n}\right]\psi_{m,n}(\theta^*) \\ & - \left(\frac{\partial \psi_{m,n}(\theta^*)}{\partial n} + \frac{\partial \psi_{m,n}(\theta^*)}{\partial \theta}\frac{\partial \theta^*}{\partial n}\right) h\left(\psi_{m,n}(\theta^*),\theta^*\right) \\ = & \left(\frac{\partial \psi_{m,n}(\theta^*)}{\partial n} + \frac{\partial \psi_{m,n}(\theta^*)}{\partial \theta}\frac{\partial \theta^*}{\partial n}\right)\left[\frac{\partial h\left(\psi_{m,n}(\theta^*),\theta^*\right)}{\partial p}\psi_{m,n}(\theta^*) - h\left(\psi_{m,n}(\theta^*),\theta^*\right)\right] \\ & + \psi_{m,n}(\theta^*)\frac{\partial h\left(\psi_{m,n}(\theta^*),\theta^*\right)}{\partial \theta}\frac{\partial \theta^*}{\partial n} \end{aligned}$$

Plugging the expression for $\frac{\partial\theta^*}{\partial n}$ into the last expression we get

$$\left(\frac{\partial\psi_{m,n}(\theta^*)}{\partial n}-\frac{\partial\psi_{m,n}(\theta^*)}{\partial\theta}\frac{\frac{\partial H}{\partial p}(\psi_{m,n}(\theta^*),\theta^*)\frac{\partial\psi_{m,n}}{\partial n}(\theta^*)}{\frac{\partial H}{\partial\theta}(\psi_{m,n}(\theta^*),\theta^*)+\frac{\partial H}{\partial p}(\psi_{m,n}(\theta^*),\theta^*)\frac{\partial\psi_{m,n}}{\partial\theta}(\theta^*)}\right)$$
$$\times\left[\frac{\partial h}{\partial p}(\psi_{m,n}(\theta^*),\theta^*)\psi_{m,n}(\theta^*)-h(\psi_{m,n}(\theta^*),\theta^*)\right]$$
$$-\psi_{m,n}(\theta^*)\frac{\partial h}{\partial\theta}(\psi_{m,n}(\theta^*),\theta^*)\frac{\frac{\partial H}{\partial p}(\psi_{m,n}(\theta^*),\theta^*)\frac{\partial\psi_{m,n}}{\partial n}(\theta^*)}{\frac{\partial H}{\partial\theta}(\psi_{m,n}(\theta^*),\theta^*)+\frac{\partial H}{\partial p}(\psi_{m,n}(\theta^*),\theta^*)\frac{\partial\psi_{m,n}}{\partial\theta}(\theta^*)}$$
$$=\frac{\partial\psi_{m,n}(\theta^*)}{\partial n}\left(1-\frac{\frac{\partial H}{\partial p}(\psi_{m,n}(\theta^*),\theta^*)\frac{\partial\psi_{m,n}}{\partial n}(\theta^*)}{\frac{\partial H}{\partial\theta}(\psi_{m,n}(\theta^*),\theta^*)+\frac{\partial H}{\partial p}(\psi_{m,n}(\theta^*),\theta^*)\frac{\partial\psi_{m,n}}{\partial\theta}(\theta^*)}\right)$$
$$\times\left[\frac{\partial h}{\partial p}(\psi_{m,n}(\theta^*),\theta^*)\psi_{m,n}(\theta^*)-h(\psi_{m,n}(\theta^*),\theta^*)\right]$$
$$-\psi_{m,n}(\theta^*)\frac{\partial h}{\partial\theta}(\psi_{m,n}(\theta^*),\theta^*)\frac{\frac{\partial H}{\partial p}(\psi_{m,n}(\theta^*),\theta^*)\frac{\partial\psi_{m,n}}{\partial n}(\theta^*)}{\frac{\partial H}{\partial\theta}(\psi_{m,n}(\theta^*),\theta^*)+\frac{\partial H}{\partial p}(\psi_{m,n}(\theta^*),\theta^*)\frac{\partial\psi_{m,n}}{\partial\theta}(\theta^*)}$$
$$=\frac{\frac{\partial\psi_{m,n}}{\partial n}(\theta^*)\frac{\partial H}{\partial\theta}(\psi_{m,n}(\theta^*),\theta^*)}{\frac{\partial H}{\partial\theta}(\psi_{m,n}(\theta^*),\theta^*)+\frac{\partial H}{\partial p}(\psi_{m,n}(\theta^*),\theta^*)\frac{\partial\psi_{m,n}}{\partial\theta}(\theta^*)}$$
$$\times\left[\frac{\partial h}{\partial p}(\psi_{m,n}(\theta^*),\theta^*)\psi_{m,n}(\theta^*)-h(\psi_{m,n}(\theta^*),\theta^*)\right]$$
$$-\psi_{m,n}(\theta^*)\frac{\partial h}{\partial\theta}(\psi_{m,n}(\theta^*),\theta^*)\frac{\frac{\partial H}{\partial p}(\psi_{m,n}(\theta^*),\theta^*)\frac{\partial\psi_{m,n}}{\partial n}(\theta^*)}{\frac{\partial H}{\partial\theta}(\psi_{m,n}(\theta^*),\theta^*)+\frac{\partial H}{\partial p}(\psi_{m,n}(\theta^*),\theta^*)\frac{\partial\psi_{m,n}}{\partial\theta}(\theta^*)}$$
$$=\frac{\frac{\partial\psi_{m,n}}{\partial n}(\theta^*)}{\frac{\partial H}{\partial\theta}(\psi_{m,n}(\theta^*),\theta^*)+\frac{\partial H}{\partial p}(\psi_{m,n}(\theta^*),\theta^*)\frac{\partial\psi_{m,n}}{\partial\theta}(\theta^*)}$$
$$\times\left[\begin{array}{c}\frac{\partial H}{\partial\theta}(\psi_{m,n}(\theta^*),\theta^*)\left(\frac{\partial h}{\partial p}(\psi_{m,n}(\theta^*),\theta^*)\psi_{m,n}(\theta^*)-h(\psi_{m,n}(\theta^*),\theta^*)\right)\\-\psi_{m,n}(\theta^*)\frac{\partial h}{\partial\theta}(\psi_{m,n}(\theta^*),\theta^*)\frac{\partial H}{\partial p}(\psi_{m,n}(\theta^*),\theta^*)\end{array}\right]$$

Since $H(0,\theta)=0$ for any θ, we have that $\frac{\partial H}{\partial\theta}(0,\theta)=0$ for any θ. Therefore $\frac{\partial^2 H}{\partial p\partial\theta}\geq 0$ implies that $\frac{\partial H}{\partial\theta}(p,\theta)\geq 0$. Recall that $\psi_{m,n}(\theta)$ is strictly decreasing in n and increasing in θ. Since H is convex in p, and $H(0,\theta)=0$ we can conclude that

$$\frac{\partial H}{\partial p}(\psi_{m,n}(\theta^*),\theta^*)\geq\frac{H(\psi_{m,n}(\theta^*),\theta^*)}{\psi_{m,n}(\theta^*)}=0$$

where the last equality follows from

$$H\left(\psi_{m,n}\left(\theta^*\right), \theta^*\right) = 0.$$

Therefore

$$\frac{\frac{\partial \psi_{m,n}}{\partial n}\left(\theta^*\right)}{\frac{\partial H}{\partial \theta}\left(\psi_{m,n}\left(\theta^*\right), \theta^*\right) + \frac{\partial H}{\partial p}\left(\psi_{m,n}\left(\theta^*\right), \theta^*\right)\frac{\partial \psi_{m,n}}{\partial \theta}\left(\theta^*\right)} < 0.$$

Hence, we need to know the sign of

$$\frac{\partial H}{\partial \theta}\left(\frac{\partial h}{\partial p}\psi_{m,n} - h\right) - \psi_{m,n}\frac{\partial h}{\partial \theta}\frac{\partial H}{\partial p}.$$

Rearranging the above expression yields

$$\frac{\partial H}{\partial \theta}\left(\frac{\partial h}{\partial p}\psi_{m,n} - h\right) - \psi_{m,n}\frac{\partial h}{\partial \theta}\frac{\partial H}{\partial p} = \psi_{m,n}\left[\frac{\partial H}{\partial \theta}\frac{\partial h}{\partial p} - \frac{\partial h}{\partial \theta}\frac{\partial H}{\partial p}\right] - h\frac{\partial H}{\partial \theta}.$$

Recall that

$$H(p,\theta) = h(p,\theta) - \frac{\partial h(p,\theta)}{\partial \theta}\frac{1-F\left(\theta\right)}{f\left(\theta\right)}$$

Therefore,

$$\begin{aligned}
\frac{\partial H(p,\theta)}{\partial \theta} &= \frac{\partial h(p,\theta)}{\partial \theta} - \frac{\partial^2 h(p,\theta)}{\partial \theta^2}\frac{1-F\left(\theta\right)}{f\left(\theta\right)} - \frac{\partial h(p,\theta)}{\partial \theta}\frac{\partial}{\partial \theta}\frac{1-F\left(\theta\right)}{f\left(\theta\right)} \\
\frac{\partial H(p,\theta)}{\partial p} &= \frac{\partial h(p,\theta)}{\partial p} - \frac{\partial^2 h(p,\theta)}{\partial \theta \partial p}\frac{1-F\left(\theta\right)}{f\left(\theta\right)}.
\end{aligned}$$

Hence

$$
\begin{aligned}
& \frac{\partial H\left(\psi_{m,n}\left(\theta^*\right),\theta^*\right)}{\partial\theta}\frac{\partial h\left(\psi_{m,n}\left(\theta^*\right),\theta^*\right)}{\partial p}-\frac{\partial h\left(\psi_{m,n}\left(\theta^*\right),\theta^*\right)}{\partial\theta}\frac{\partial H\left(\psi_{m,n}\left(\theta^*\right),\theta^*\right)}{\partial p} \\
= & \frac{\partial h\left(\psi_{m,n}\left(\theta^*\right),\theta^*\right)}{\partial\theta}\frac{\partial h\left(\psi_{m,n}\left(\theta^*\right),\theta^*\right)}{\partial p}-\frac{\partial h\left(\psi_{m,n}\left(\theta^*\right),\theta^*\right)}{\partial p}\frac{\partial^2 h\left(\psi_{m,n}\left(\theta^*\right),\theta^*\right)}{\partial\theta^2}\frac{1-F\left(\theta^*\right)}{f\left(\theta^*\right)} \\
& -\frac{\partial h\left(\psi_{m,n}\left(\theta^*\right),\theta^*\right)}{\partial p}\frac{\partial h\left(\psi_{m,n}\left(\theta^*\right),\theta^*\right)}{\partial\theta}\frac{\partial}{\partial\theta}\frac{1-F\left(\theta^*\right)}{f\left(\theta^*\right)} \\
& -\frac{\partial h\left(\psi_{m,n}\left(\theta^*\right),\theta^*\right)}{\partial p}\frac{\partial h\left(\psi_{m,n}\left(\theta^*\right),\theta^*\right)}{\partial\theta}+\frac{\partial h\left(\psi_{m,n}\left(\theta^*\right),\theta^*\right)}{\partial\theta}\frac{\partial^2 h\left(\psi_{m,n}\left(\theta^*\right),\theta^*\right)}{\partial\theta\partial p}\frac{1-F\left(\theta^*\right)}{f\left(\theta^*\right)} \\
= & -\frac{\partial h\left(\psi_{m,n}\left(\theta^*\right),\theta^*\right)}{\partial p}\frac{\partial^2 h\left(\psi_{m,n}\left(\theta^*\right),\theta^*\right)}{\partial\theta^2}\frac{1-F\left(\theta^*\right)}{f\left(\theta^*\right)} \\
& \qquad -\frac{\partial h\left(\psi_{m,n}\left(\theta^*\right),\theta^*\right)}{\partial p}\frac{\partial h\left(\psi_{m,n}\left(\theta^*\right),\theta^*\right)}{\partial\theta}\frac{\partial}{\partial\theta}\frac{1-F\left(\theta^*\right)}{f\left(\theta^*\right)} \\
& \qquad\quad +\frac{\partial h\left(\psi_{m,n}\left(\theta^*\right),\theta^*\right)}{\partial\theta}\frac{\partial^2 h\left(\psi_{m,n}\left(\theta^*\right),\theta^*\right)}{\partial\theta\partial p}\frac{1-F\left(\theta^*\right)}{f\left(\theta^*\right)}.
\end{aligned}
$$

From the definition of the optimal reserve price we obtain that

$$h\left(\psi_{m,n}\left(\theta^*\right),\theta^*\right)=\frac{\partial h\left(\psi_{m,n}\left(\theta^*\right),\theta^*\right)}{\partial\theta}\frac{1-F\left(\theta^*\right)}{f\left(\theta^*\right)}$$

(since θ^* solves $H\left(\psi_{m,n}\left(\theta^*\right),\theta^*\right)=0$). Therefore,

$$\begin{aligned}
& \psi_{m,n}\left(\theta^*\right)\left[\frac{\partial H\left(\psi_{m,n}\left(\theta^*\right),\theta^*\right)}{\partial\theta}\frac{\partial h\left(\psi_{m,n}\left(\theta^*\right),\theta^*\right)}{\partial p}-\frac{\partial h\left(\psi_{m,n}\left(\theta^*\right),\theta^*\right)}{\partial\theta}\frac{\partial H\left(\psi_{m,n}\left(\theta^*\right),\theta^*\right)}{\partial p}\right] \\
& \qquad -h\left(\psi_{m,n}\left(\theta^*\right),\theta^*\right)\frac{\partial H\left(\psi_{m,n}\left(\theta^*\right),\theta^*\right)}{\partial\theta} \\
= \ & \psi_{m,n}\left(\theta^*\right)\left[\begin{array}{c} -\frac{\partial h(\psi_{m,n}(\theta^*),\theta^*)}{\partial p}\frac{\partial^2 h(\psi_{m,n}(\theta^*),\theta^*)}{\partial\theta^2}\frac{1-F(\theta^*)}{f(\theta^*)}-\frac{\partial h(\psi_{m,n}(\theta^*),\theta^*)}{\partial p}\frac{\partial h(\psi_{m,n}(\theta^*),\theta^*)}{\partial\theta}\frac{\partial}{\partial\theta}\frac{1-F(\theta^*)}{f(\theta^*)} \\ +\frac{\partial h(\psi_{m,n}(\theta^*),\theta^*)}{\partial\theta}\frac{\partial^2 h(\psi_{m,n}(\theta^*),\theta^*)}{\partial\theta\partial p}\frac{1-F(\theta^*)}{f(\theta^*)} \end{array}\right] \\
& -h_\theta\left(\psi_{m,n}\left(\theta^*\right),\theta^*\right)\frac{1-F\left(\theta^*\right)}{f\left(\theta^*\right)}\left[\frac{\partial h\left(\psi_{m,n}\left(\theta^*\right),\theta^*\right)}{\partial\theta}-\frac{\partial^2 h\left(\psi_{m,n}\left(\theta^*\right),\theta^*\right)}{\partial\theta^2}\frac{1-F\left(\theta^*\right)}{f\left(\theta^*\right)}\right. \\
& \qquad \left.-\frac{\partial h\left(\psi_{m,n}\left(\theta^*\right),\theta^*\right)}{\partial\theta}\frac{\partial}{\partial\theta}\frac{1-F\left(\theta^*\right)}{f\left(\theta^*\right)}\right] \\
= \ & -\frac{\partial}{\partial\theta}\frac{1-F\left(\theta^*\right)}{f\left(\theta^*\right)}\left[\psi_{m,n}\left(\theta^*\right)\frac{\partial h\left(\psi_{m,n}\left(\theta^*\right),\theta^*\right)}{\partial p}\frac{\partial h\left(\psi_{m,n}\left(\theta^*\right),\theta^*\right)}{\partial\theta}\right. \\
& \qquad \left.+\left(\frac{\partial h\left(\psi_{m,n}\left(\theta^*\right),\theta^*\right)}{\partial\theta}\right)^2\frac{1-F\left(\theta^*\right)}{f\left(\theta^*\right)}\right] \\
& +\frac{1-F\left(\theta^*\right)}{f\left(\theta^*\right)}\left[\begin{array}{c} -\psi_{m,n}\left(\theta^*\right)\frac{\partial h(\psi_{m,n}(\theta^*),\theta^*)}{\partial p}\frac{\partial^2 h(\psi_{m,n}(\theta^*),\theta^*)}{\partial\theta^2}+\psi_{m,n}\left(\theta^*\right)\frac{\partial h(\psi_{m,n}(\theta^*),\theta^*)}{\partial\theta}\frac{\partial^2 h(\psi_{m,n}(\theta^*),\theta^*)}{\partial\theta\partial p} \\ -\left(\frac{\partial h(\psi_{m,n}(\theta^*),\theta^*)}{\partial\theta}\right)^2+\frac{\partial h(\psi_{m,n}(\theta^*),\theta^*)}{\partial\theta}\frac{\partial^2 h(\psi_{m,n}(\theta^*),\theta^*)}{\partial\theta^2}\frac{1-F(\theta^*)}{f(\theta^*)} \end{array}\right]
\end{aligned}$$

The assumptions $\frac{\partial^2 h}{\partial\theta^2}\leq 0$, and $\frac{\partial}{\partial\theta}(\frac{1-F}{f})\leq -1$ guarantee that the above expression is positive. Since

$$\frac{\partial}{\partial\theta}(\frac{1-F\left(\theta^*\right)}{f\left(\theta^*\right)})=\frac{-\left(f\left(\theta^*\right)\right)^2-f'\left(\theta^*\right)F\left(\theta^*\right)}{\left(f\left(\theta^*\right)\right)^2}=-1-\frac{f'\left(\theta^*\right)F\left(\theta^*\right)}{\left(f\left(\theta^*\right)\right)^2},$$

we obtain that $f'\left(\theta^*\right)\geq 0$ is sufficient for $\frac{\partial R^*}{\partial n}<0$. The proof that $\frac{\partial R^*}{\partial m}>0$ is similar, the only difference being that $\psi_{m,n}(\theta)$ is increasing in m. □

Proof of Lemma 3: The expected revenue is given by

$$R=n\int_{\theta^*}^{1}(p)^{\frac{l}{l-1}}\left(\frac{\theta}{b}\right)^{\frac{1}{l-1}}K(\theta)f\left(\theta\right)d\theta$$

where $\frac{l-1}{l}\theta^*-\frac{1-F(\theta^*)}{f(\theta^*)}=0$ and where $p=F^{n-1}\left(\theta\right)$.

The derivative of the expected revenue with respect to n is

$$\int_{\theta^*}^{1} F^{\frac{(n-1)l}{l-1}}(\theta)\left(\frac{\theta}{b}\right)^{\frac{1}{l-1}} K(\theta) f(\theta)\, d\theta + n\int_{\theta^*}^{1} F^{\frac{(n-1)l}{l-1}}(\theta)\frac{l}{l-1}\ln F(\theta)\left(\frac{\theta}{b}\right)^{\frac{1}{l-1}} K(\theta) f(\theta)\, d\theta$$
$$= \int_{\theta^*}^{1} K(\theta)\left(\frac{\theta}{b}\right)^{\frac{1}{l-1}} F^{\frac{(n-1)l}{l-1}}(\theta)\left[1+n\frac{l}{l-1}\ln F(\theta)\right] f(\theta)\, d\theta$$

This can be written as:

$$\int_{\theta^*}^{1} K(\theta)\frac{1}{F^{\frac{1}{l-1}}(\theta)}\left(\frac{\theta}{b}\right)^{\frac{1}{l-1}} F^{\frac{(n-1)l}{l-1}}(\theta)\left[1+n\frac{l}{l-1}\ln F(\theta)\right] F^{\frac{1}{l-1}}(\theta)\ f(\theta)\, d\theta$$

Because $1+n\frac{l}{l-1}\ln F(\theta)$ changes sign only once as a function of θ , from negative to positive, and by the concavity of F and monotonicity of K, the function $K(\theta)\left(\frac{\theta}{F(\theta)}\right)^{\frac{1}{l-1}}$ is positive and increasing. Thus, it is sufficient to show that[33]

$$\int_{\theta^*}^{1} F^{\frac{(n-1)l}{l-1}}(\theta)\left[1+n\frac{l}{l-1}\ln F(\theta)\right] F^{\frac{1}{l-1}}(\theta)\ f(\theta)\, d\theta \geq 0.$$

Define a distribution $S(\theta) = F^{\frac{l}{l-1}}(\theta)$ and let $s(\theta) = \frac{l}{l-1}F^{\frac{1}{l-1}}(\theta)$ be its density. Using the new notation we have

$$\int_{\theta^*}^{1} S^{n-1}(\theta)\left[1+n\ln S(\theta)\right] s(\theta)\, d\theta = \int_{\theta^*}^{1} S^{n-1}(\theta)\, s(\theta)\, d\theta + \int_{\theta^*}^{1} S^{n-1}(\theta)\, n\ln S(\theta)\, s(\theta)\, d\theta$$
$$= \int_{\theta^*}^{1} S^{n-1}(\theta)\, s(\theta)\, d\theta + S^{n}(\theta)\ln S(\theta)\,|_{\theta=\theta^*}^{1} - \int_{\theta^*}^{1} S^{n}(\theta)\frac{s(\theta)}{S(\theta)}d\theta = -S^{n}(\theta^*)\ln S(\theta^*) > 0\,. \quad \square$$

Example Split Awards:

1. Single award.

$$h(p,\theta) = \max_{a} p2(a+\theta) - b\frac{a^2}{2}$$

The optimal investment is $a = \frac{2p}{b}$ and hence

[33] Assume $k(\cdot)$ changes sign once at $x^* \in [a,b]$ from negative to positive and assume that $d(\cdot)$ is positive increasing on the interval $[a,b]$. Then $\int_a^b k(t)dt \geq 0$ implies

$$\int_a^b k(t)d(t)dt = \int_a^x k(t)d(t)dt + \int_x^b k(t)d(t)dt \geq d(x)\int_a^x k(t)dt + d(x)\int_x^b k(t)dt = d(x)\int_a^b k(t)dt \geq 0$$

$$h(p,\theta) = p2\left(\frac{2p}{b}+\theta\right) - \frac{b}{2}\left(\frac{2p}{b}\right)^2 = 2\frac{p^2}{b} + 2p\theta$$

and

$$H(p,\theta) = h(p,\theta) - \frac{\partial h}{\partial \theta}\frac{1-F(\theta)}{f(\theta)} = 2p\left(\theta - \frac{1-F(\theta)}{f(\theta)} + \frac{p}{b}\right)$$

Therefore, the expected revenue in the optimal mechanism with single award is

$$\begin{aligned} & n\int_{\theta^*}^{1} 2F^{n-1}(\theta)\left(\theta - \frac{1-F(\theta)}{f(\theta)} + \frac{F^{n-1}(\theta)}{b}\right) f(\theta)\,d\theta \\ = \; & n\int_{\theta^*}^{1} 2F^{n-1}(\theta)\left(\theta - \frac{1-F(\theta)}{f(\theta)} + \frac{2F^{n-1}(\theta)}{2b}\right) f(\theta)\,d\theta \end{aligned} \tag{29}$$

where θ^* is the solution to

$$\theta - \frac{1-F(\theta)}{f(\theta)} + \frac{F^{n-1}(\theta)}{b} = 0$$

2. Split award.

In this case

$$h(p,\theta) = \max_a p(a+\theta) - b\frac{a^2}{2}$$

The optimal investment is $a = \frac{p}{b}$ and then

$$h(p,\theta) = p\left(\frac{p}{b}+\theta\right) - \frac{b}{2}\left(\frac{p}{b}\right)^2 = \frac{p^2}{2b} + p\theta$$

and

$$H(p,\theta) = h(p,\theta) - \frac{\partial h}{\partial \theta}\frac{1-F(\theta)}{f(\theta)} = p\left(\theta - \frac{1-F(\theta)}{f(\theta)} + \frac{p}{2b}\right)$$

In the optimal mechanism $p = F^{n-1}(\theta) + nF^{n-2}(\theta)(1-F(\theta))$. Therefore, the expected revenue in the optimal mechanism with split award is

$$\begin{aligned} & n\int_{\theta^{**}}^{1} \left[F^{n-1}(\theta) + nF^{n-2}(\theta)(1-F(\theta))\right] \\ & \qquad \times \left(\theta - \frac{1-F(\theta)}{f(\theta)} + \frac{F^{n-1}(\theta) + nF^{n-2}(\theta)(1-F(\theta))}{2b}\right) f(\theta)\,d\theta \end{aligned}$$

where now θ^{**} is the solution to

$$\theta - \frac{1-F(\theta)}{f(\theta)} + \frac{F^{n-1}(\theta) + nF^{n-2}(\theta)(1-F(\theta))}{2b} = 0$$

We analyze below two limit cases that yield contrasting results:

Case 1: $b \to \infty$. Here costs become prohibitively high. Then

$$\lim_{b\to\infty} \theta^{**} = \lim_{b\to\infty} \theta^{*} = \hat{\theta}$$

where $\hat{\theta}$ solves

$$\theta - \frac{1-F(\theta)}{f(\theta)} = 0.$$

The expected revenue from of a single award is

$$n \int_{\hat{\theta}}^{1} 2F^{n-1}(\theta) \left(\theta - \frac{1-F(\theta)}{f(\theta)} \right) f(\theta)\, d\theta$$

and from a split award is given by

$$n \int_{\hat{\theta}}^{1} \left[F^{n-1}(\theta) + nF^{n-2}(\theta)(1-F(\theta)) \right] \left(\theta - \frac{1-F(\theta)}{f(\theta)} \right) f(\theta)\, d\theta$$

Observe that, if the virtual value is monotonic, a single award generates higher expected revenue: the split award corresponds to a random mechanism that allocates the object with equal probabilities between the two highest bidders, which is suboptimal.

Case 2. $b \to 0$. Here costs are very low . In this case

$$\lim_{b\to\infty} \theta^{**} = \lim_{b\to\infty} \theta^{*} = 0$$

and the expected revenue from a single award is

$$\lim_{b\to 0} n \int_{\theta^{*}}^{1} 2F^{n-1}(\theta) \left(\theta - \frac{1-F(\theta)}{f(\theta)} + \frac{2F^{n-1}(\theta)}{2b} \right) f(\theta)\, d\theta = \lim_{b\to 0} n \int_{0}^{1} 2F^{n-1}(\theta) \frac{2F^{n-1}(\theta)}{2b} f(\theta)\, d\theta$$

while the expected revenue from splitting the award is

$$\begin{aligned} & \lim_{b\to 0} n \int_{\theta^{**}}^{1} \left[F^{n-1}(\theta) + nF^{n-2}(\theta)(1-F(\theta)) \right] \\ & \qquad \times \left(\theta - \frac{1-F(\theta)}{f(\theta)} + \frac{F^{n-1}(\theta) + nF^{n-2}(\theta)(1-F(\theta))}{2b} \right) f(\theta)\, d\theta \\ = & \lim_{b\to 0} n \int_{0}^{1} \left[F^{n-1}(\theta) + nF^{n-2}(\theta)(1-F(\theta)) \right] \left(\frac{F^{n-1}(\theta) + nF^{n-2}(\theta)(1-F(\theta))}{2b} \right) f(\theta)\, d\theta \end{aligned}$$

Denoting $M(\theta) = 2F^{n-1}(\theta)$ and $P(\theta) = F^{n-1}(\theta) + nF^{n-2}(\theta)(1 - F(\theta))$, the difference in the expected revenues is given by

$$\lim_{b \to 0} \frac{n}{2b} \int_0^1 \left[M^2(\theta) - P^2(\theta)\right] f(\theta)\, d\theta$$

Hence splitting the award generates higher expected revenue if and only if

$$\int_0^1 \left[M^2(\theta) - P^2(\theta)\right] f(\theta)\, d\theta > 0$$

Note that

$$\begin{aligned} M^2 - P^2 &= 4F^{2n-2} - F^{2n-2} - 2nF^{2n-3}(1-F) - n^2F^{2n-4}(1-F)^2 \\ &= \left(3 + 2n - n^2\right) F^{2n-2} - n^2F^{2n-4} + 2n(n-1)F^{2n-3} \end{aligned}$$

Therefore,

$$\begin{aligned} &\int_0^1 \left[M^2(\theta) - P^2(\theta)\right] f(\theta)\, d\theta \\ &= \int_0^1 \left[\left(3 + 2n - n^2\right) F^{2n-2}(\theta) - n^2F^{2n-4}(\theta) + 2n(n-1)F^{2n-3}(\theta)\right] f(\theta)\, d\theta \\ &= \frac{3(n-3)}{4n^2 - 8n + 3}. \end{aligned}$$

Proof of Lemma 4. Assume, by contradiction, that the statement is not correct. Hence there exists an interval $[t', t'']$ such that the extreme point p is continuous and strictly increasing on this interval, but $p(t) \neq \phi_{m,n}(t)$.

We shall exhibit a function $u(t) \neq 0$ such that both functions $p + u$ and $p - u$ are non-decreasing and satisfy

$$p + u \prec_w \phi_{m,n} \text{ and } p - u \prec_w \phi_{m,n}$$

contradicting the hypothesis that p is extreme.

By possibly choosing a subset of $[t', t'']$, we can assume without loss of generality that either $p(t) < \phi_{m,n}(t)$ for $t \in [t', t'']$ or $p(t) > \phi_{m,n}(t)$ for $t \in [t', t'']$, and we focus here on the first possibility (the latter is completely analogous).

Choose t_1, t_2, t_3 such that $t' < t_1 < t_2 < t_3 < t''$. Denote by $\delta_1 = \min_{t \in [t_1, t_2]} p'(t)$ (if the derivative does not exist we take the minimum between the left and right derivatives that always exist since p is monotonic), and similarly denote $\delta_2 = \min_{t\in[t_2,t_3]} p'(t)$. By assumptions, $\delta_1 > 0$ and $\delta_2 > 0$. We define u as follows:

$$u(t) = \begin{cases} \delta_1 (t - t_1) & \text{if } t \in [t_1, t_1 + \epsilon_1] \\ \delta_1\epsilon_1 - \delta_1 (t - t_1 - \epsilon_1) & \text{if } t \in [t_1 + \epsilon_1, t_1 + 2\epsilon_1] \\ -\delta_2 (t - t_3 + 2\epsilon_2) & \text{if } t \in [t_3 - 2\epsilon_2, t_3 - \epsilon_2] \\ -\delta_2\epsilon_2 + \delta_2 (t - t_3 + \epsilon_2) & \text{if } t \in [t_3 - \epsilon_2, t_3] \\ 0 & \text{otherwise} \end{cases}$$

where ϵ_1 and ϵ_2 are chosen such that

$$\begin{aligned} (1)\ \epsilon_1 &\leq t_2 - t_1, \epsilon_2 \leq t_3 - t_2, \\ (2)\ \delta_1\epsilon_1 - \delta_2\epsilon_2 &= 0 \\ (3)\ p(t) - \delta_2\epsilon_2 &< \phi_{m,n}(t)\ ,\ t \in [t_2, t_3]\,. \end{aligned}$$

Next Figure illustrates it.

The second condition guarantees that $\int u(t)\,dt = \delta_1\epsilon_1 - \delta_2\epsilon_2 = 0$. Notice that, by construction, both $p+u$ and $p-u$ are monotone. Moreover by construction $p+u \prec \phi_{m,n}(t)$, while condition (3) guarantees that $p-u \prec \phi_{m,n}(t)$. □

8.1 Sufficient conditions for CSM Environments

When does our "virtual utility" function

$$H(p,\theta) = h(p,\theta) - h_\theta(p,\theta) \times \frac{1-F(\theta)}{f(\theta)}$$

satisfies the conditions in the Fan-Lorentz Theorem, i.e. when is the environment convex super-modular? We have

$$\frac{\partial^2 H}{(\partial p)^2} = \frac{\partial^2 h}{(\partial p)^2} - \frac{\partial^3 h}{\partial\theta(\partial p)^2}(\frac{1-F(\theta)}{f(\theta)})$$

Since $\frac{\partial^2 h}{(\partial p)^2} \geq 0$ by assumption and because $\frac{1-F(\theta)}{f(\theta)} \geq 0$, a sufficient condition for $\frac{\partial^2 H}{(\partial p)^2} \geq 0$ is $\frac{\partial^3 h}{\partial\theta(\partial p)^2} \leq 0$.

We also have

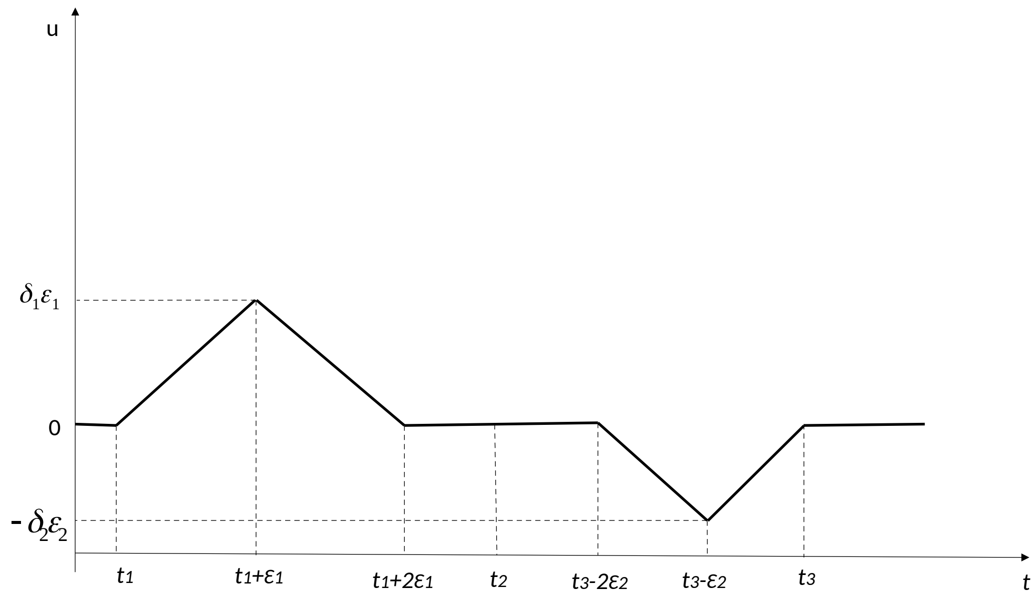

Figure 3: function u

$$\begin{aligned}\frac{\partial^2 H}{\partial\theta\partial p} &= \frac{\partial^2 h}{\partial\theta\partial p} - \frac{\partial^2 h}{\partial\theta\partial p}\frac{d}{d\theta}(\frac{1-F(\theta)}{f(\theta)}) - \frac{\partial^3 h}{(\partial\theta)^2\partial p}(\frac{1-F(\theta)}{f(\theta)}) = \\ & \frac{\partial^2 h}{\partial\theta\partial p}(1 - \frac{d}{d\theta}(\frac{1-F(\theta)}{f(\theta)})) - \frac{\partial^3 h}{(\partial\theta)^2\partial p}(\frac{1-F(\theta)}{f(\theta)}).\end{aligned}$$

A sufficient conditions for $\frac{\partial^2 H}{\partial\theta\partial p} \geq 0$ is an increasing hazard rate, i.e. $\frac{d}{d\theta}(\frac{f(\theta)}{1-F(\theta)}) \geq 0$ and $\frac{\partial^3 h}{(\partial\theta)^2\partial p} \leq 0$. To conclude sufficient conditions are :

$$\begin{aligned}1.\ \frac{\partial^3 h}{\partial\theta(\partial p)^2} &\leq 0, \frac{\partial^3 h}{(\partial\theta)^2\partial p} \leq 0 \\ & 2.\ IFR\end{aligned}$$

These conditions can be further decomposed into standard sufficient conditions on the functions v and c.

The above derivations can be compared to the classical treatment, e.g., see Fudenberg and Tirole (1991) (Section 7.3.2). They only consider a single agent and hence need not consider the resource constraint that is our main concern here. Their sufficient conditions are:

$$
\begin{aligned}
FT1.\ \frac{\partial^3 h}{\partial\theta(\partial p)^2} &\geq 0, \frac{\partial^3 h}{(\partial\theta)^2\partial p} \leq 0 \\
& FT2.\ IFR
\end{aligned}
$$

The difference to our conditions is due to their assumption that the valuation is **concave** in the allocation. Thus, they initially assume $\frac{\partial^3 h}{(\partial p)^2} \leq 0$, and having $\frac{\partial^3 h}{\partial\theta(\partial p)^2} \geq 0$ ensures that their welfare function is concave so that the First Order Approach can be employed.

8.2 Illustration for Other Goals

Our insights can be used to identify the optimal mechanisms in situations where the goal is different from revenue maximization. For example, Chawla, Hartline and Sivan (2015) study an auction for an indivisible object where the goal of the designer is to maximize the highest bid (rather than the sum of bids as in the optimal auction a la Myerson). They show that, when restricted to symmetric mechanisms, their maximization problem can be written as

$$n \int_{\Theta_i} p(\theta) N(\theta) f(\theta) d\theta$$

where the "virtual utility" $N : [0, \overline{\theta}] \to \mathbb{R}$ is defined by

$$N(\theta) := \theta F(\theta)^{n-1} - \frac{1 - F^n(\theta)}{n f(\theta)}$$

Because of the linear, separable form in p, it is again clear that the Fan-Lorentz conditions are satisfied if N is non-decreasing, and that the optimal mechanism is a standard auction with a reserve price. Note that, the optimal cutoff $N^{-1}(0)$ is depending on the number of bidders n. For example, taking the uniform distribution on $[0, 1]$ yields

$$N(\theta) = \theta^n \left(1 + \frac{1}{n}\right) - \frac{1}{n}$$

and the optimal cutoff is

$$\theta_n^* = \left(\frac{1}{n+1}\right)^{-\frac{1}{n}}$$

which is decreasing here in n.