# Necessity of Fully Substitutable Preferences

*Online Appendix to*

"Stability and Competitive Equilibrium in Trading Networks"


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**Abstract**

In this appendix, we provide a proof of the necessity result (Theorem 7) of "Stability and Competitive Equilibrium in Trading Networks."



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**Theorem 7.** *Suppose that there exist at least four agents and that the set of trades is exhaustive. Then if the preferences of some agent are not fully substitutable, there exist simple preferences for all other agents such that no stable outcome exists.*

*Proof.* Suppose that the preferences of $i$ are not fully substitutable, and in particular fail part (1) of the definition of (DFS) with unique demands (Definition 4). Then there exist price vectors $p$ and $p'$ and trades $\omega$ and $\psi$ where $b(\omega) = i$ and $p'_{-\omega} = p_{-\omega}, p'_\omega > p_\omega$ and $\{\Psi\} = D_i(p)$ and $\{\Psi'\} = D_i(p')$ and either

**Case 1** $b(\psi) = i$ and $\psi \in \Psi$ but $\psi \notin \Psi'$, or

**Case 2** $s(\psi) = i$ and $\psi \notin \Psi$ but $\psi \in \Psi'$.

For every trade $\xi \in \Omega_i - \{\psi, \omega\}$, if $b(\xi) = i$ we let

$$u_{s(\xi)}(\Xi - \{\xi\}) - u_{s(\xi)}(\Xi \cup \{\xi\}) = p_\xi$$

for all $\Xi \subseteq \Omega$ and similarly, if $s(\xi) = i$, then we let

$$u_{b(\xi)}(\Xi \cup \{\xi\}) - u_{b(\xi)}(\Xi - \{\xi\}) = p_\xi$$

for all $\Xi \subseteq \Omega$. It is clear that these preferences, restricted to $\Omega_i - \{\psi, \omega\}$, are simple for every agent.

Without further specification of agents' preferences we can infer that whenever a stable outcome $A$ exists, the outcome

$$\bar{A} = (A - \{(\xi, q_\xi) : \xi \in [\tau(A_i) - \{\psi, \omega\}] \text{ and } (\xi, q_\xi) \in A\}) \cup \{(\xi, p_\xi) : \xi \in [\tau(A_i) - \{\psi, \omega\}]\}$$

will also be stable. To see this, note that if $(\xi, q_\xi) \in A$ for some $\xi \neq \psi, \omega$ such that $b(\xi) = i$ then $q_\xi \geq p_\xi$ due to the preferences defined above. If $q_\xi > p_\xi$ then $\tilde{A} \equiv [A - (\xi, q_\xi)] \cup \{(\xi, p_\xi)\}$ is also a stable match, as it is clearly individually rational as $A$ is individually rational, and if $Z$ was a blocking set for $\tilde{A}$, it would also be a blocking set for $A$. Similarly, if $(\xi, q_\xi) \in A$, $s(\xi) = i$, and $\xi \neq \psi, \omega$ then $q_\xi \leq p_\xi$ and so $[A - (\xi, q_\xi)] \cup \{(\xi, p_\xi)\}$ is also a stable match. The above statement now follows by induction and we will use it repeatedly in our proof below.

It will be helpful to define the marginal utility agent $i$ obtains from having available trades in some set $\Phi \subseteq \{\psi, \omega\}$ in addition to having trades in $\Omega_i - \{\psi, \omega\}$ at their prices according to the price vector $p$ by

$$v_i(\Phi) \equiv \max_{\substack{\Xi \subseteq \Omega_i - \{\psi, \omega\} \\ \hat{\Phi} \subseteq \Phi}} \left\{ u_i\left(\Xi \cup \hat{\Phi}\right) + \sum_{\xi \in \Xi_{i\to}} p_\xi - \sum_{\xi \in \Xi_{\to i}} p_\xi \right\}.$$

We now proceed to discuss the two possible cases.

**Case 1** $b(\psi) = i$ and $\psi \in \Psi$ but $\psi \notin \Psi'$: Note that

$$v_i(\{\psi, \omega\}) - v_i(\{\omega\}) > v_i(\{\psi\}) - v_i(\varnothing) \geq 0$$

as otherwise we would have that $\psi \in \Psi'$, as if

$$v_i(\{\psi,\omega\}) - v_i(\{\omega\}) \le v_i(\{\psi\}) - v_i(\varnothing)$$

then $i$ must demand $\psi$ at prices $(p_{-\omega}, p'_\omega)$ as $i$ demanded $\psi$ at prices $p$.

Now let $\hat{\psi}, \hat{\omega}$ be two trades such that $s(\psi) = s\left(\hat{\psi}\right)$, $s(\omega) = s(\hat{\omega})$, and $b\left(\hat{\psi}\right) = b(\hat{\omega}) = j \neq i$ (such a trade must exist as there are at least four agents and the set of trades is exhaustive). Let $s(\psi), s(\omega)$ have preferences such that

$$u_{s(\psi)}(\Xi \cup \{\psi\}) - u_{s(\psi)}(\Xi) = u_{s(\psi)}\left(\Xi \cup \left\{\hat{\psi}\right\}\right) - u_{s(\psi)}(\Xi) = 0$$
$$u_{s(\psi)}\left(\Xi \cup \left\{\psi, \hat{\psi}\right\}\right) = -\infty$$

for all $\Xi \subseteq \Omega - \left\{\psi, \hat{\psi}\right\}$ and

$$u_{s(\omega)}(\Xi \cup \{\omega\}) - u_{s(\omega)}(\Xi) = u_{s(\omega)}(\Xi \cup \{\hat{\omega}\}) - u_{s(\omega)}(\Xi) = 0$$
$$u_{s(\omega)}(\Xi \cup \{\omega, \hat{\omega}\}) = -\infty$$

for all $\Xi \subseteq \Omega - \{\omega, \hat{\omega}\}$. It is possible that $s(\psi) = s(\omega)$.

Let $j$'s preferences satisfy

$$u_j\left(\left\{\hat{\psi}\right\} \cup \Xi\right) - u_j(\Xi) = \frac{2\left[v_i(\{\psi,\omega\}) - v_i(\{\omega\})\right] + \left[v_i(\{\psi\}) - v_i(\varnothing)\right]}{3} \equiv t(\psi)$$
$$u_j(\{\hat{\omega}\} \cup \Xi) - u_j(\Xi) = \frac{2\left[v_i(\{\psi,\omega\}) - v_i(\{\psi\})\right] + \left[v_i(\{\omega\}) - v_i(\varnothing)\right]}{3} \equiv t(\omega)$$
$$u_j\left(\left\{\hat{\psi}, \hat{\omega}\right\} \cup \Xi\right) - u_j(\Xi) = -\infty$$

for all $\Xi \subseteq \Omega - \left\{\hat{\psi}, \hat{\omega}\right\}$. Note that due to the above inequality we must have

$$0 < t(\psi) < v_i(\{\psi,\omega\}) - v_i(\{\omega\})$$

and

$$0 < t(\omega) < v_i(\{\psi,\omega\}) - v_i(\{\psi\}).$$

Clearly, the preferences of all agents but $i$ can be extended to simple preferences on $\Omega$.[1] There are four subcases to consider to show that $\bar{A}$ can not be stable:

1. $\tau\left(\bar{A}\right) \cap \{\psi, \omega\} = \emptyset$: If both $\hat{\psi}$ and $\hat{\omega} \in \tau\left(\bar{A}\right)$, then $\bar{A}$ is not individually rational for $j$. If $\hat{\psi}, \hat{\omega} \notin \tau\left(\bar{A}\right)$, then $\left\{\left(\hat{\psi}, \epsilon\right)\right\}$ is a block. Hence, exactly one of $\hat{\psi}$ and $\hat{\omega}$ is in $\tau\left(\bar{A}\right)$. Suppose $\left(\hat{\psi}, q_{\hat{\psi}}\right) \in \bar{A}$ for some $q_{\hat{\psi}} \in \mathbb{R}_+$. Individual rationality for $j$ requires that

$$q_{\hat{\psi}} \le t(\psi) < v_i(\{\psi,\omega\}) - v_i(\{\omega\}).$$

[1]Throughout the proof we only specify the parts of the preferences that are important to show that no stable outcome can exist.

But then

$$Z \equiv \left\{\left(\xi, p_\xi + \epsilon \mathbb{I}_{b(\xi)=i} - \epsilon \mathbb{I}_{s(\xi)=i}\right) : \xi \in \Psi - \tau\left(\bar{A}_i\right)\right\} \cup \left\{\left(\psi, q_{\hat{\psi}} + \epsilon\right), (\omega, \epsilon)\right\}$$

is a blocking set for some small $\epsilon > 0$. Note that $(\omega, \epsilon)$ strictly increases by $\epsilon$ the utility of $s(\omega)$, no matter what other contracts $s(\omega)$ chooses. Similarly, for all $\xi \in \Psi - \tau\left(\bar{A}\right)$, $\left(\xi, p_\xi + \epsilon \mathbb{I}_{b(\xi)=i} - \epsilon \mathbb{I}_{s(\xi)=i}\right)$ strictly increases by $\epsilon$ the utility of the agent other than $i$ associated with this contract, no matter what other contracts that agent chooses. Agent $s\left(\hat{\psi}\right)$ will choose contract $\left(\psi, q_{\hat{\psi}} + \epsilon\right)$ and not choose $\left(\hat{\psi}, q_{\hat{\psi}}\right)$, regardless of other contracts he chooses. Finally, agent $i$'s choice $\bar{A} \cup Z$ is single valued and includes $Z$, as the above inequality implies that if $i$ chooses $(\omega, \epsilon)$, he must also choose $\left(\psi, q_{\hat{\psi}} + \epsilon\right)$. We also have that $v_i(\{\omega\}) \geq v_i(\varnothing)$, implying that for $\epsilon$ small enough $i$ will choose both $\left(\psi, q_{\hat{\psi}} + \epsilon\right)$ and $(\omega, \epsilon)$ from $\bar{A} \cup Z$, and hence $i$ will choose all of the contracts associated with trades in $\Psi$ as such contracts are optimal at prices $p$.

If $(\hat{\omega}, q_{\hat{\omega}}) \in \bar{A}$ for some $q_{\hat{\omega}} \in \mathbb{R}_+$, we obtain a similar contradiction since individual rationality for $j$ requires that $q_{\hat{\omega}} \leq t(\omega) < v_i(\{\psi, \omega\}) - v_i(\{\psi\})$.

2. $(\psi, q_\psi) \in \bar{A}$ for some $q_\psi \in \mathbb{R}_+$ and $\omega \notin \tau\left(\bar{A}\right)$: In this case we must have $(\hat{\omega}, q_{\hat{\omega}}) \in \bar{A}$ for some $q_{\hat{\omega}} \in \mathbb{R}_+$, as otherwise $\{(\hat{\omega}, \epsilon)\}$ for some small $\epsilon > 0$ would be a blocking set since $j$'s incremental utility of signing $\hat{\omega}$ is $t(\omega) > 0$. Individual rationality for $j$ requires

$$q_{\hat{\omega}} \leq t(\omega) < v_i(\{\psi, \omega\}) - v_i(\{\omega\}).$$

Furthermore, we must have

$$q_\psi \leq v_i(\{\psi\}) - v_i(\varnothing)$$

as otherwise either $\bar{A}$ is not individually rational for $i$ or

$$\left\{\left(\xi, p_\xi + \epsilon \mathbb{I}_{b(\xi)=i} - \epsilon \mathbb{I}_{s(\xi)=i}\right) : \xi \in \Psi' - \tau\left(\bar{A}\right)\right\}$$

is a blocking set for $\epsilon > 0$ sufficiently small. However, these inequalities imply that

$$\left\{\left(\xi, p_\xi + \epsilon \mathbb{I}_{b(\xi)=i} - \epsilon \mathbb{I}_{s(\xi)=i}\right) : \xi \in \Psi - \tau\left(\bar{A}\right)\right\} \cup \{(\omega, p_{\hat{\omega}} + \epsilon)\}$$

is a blocking set for some small $\epsilon > 0$.

3. $(\omega, q_\omega) \in \bar{A}$ for some $q_\omega \in \mathbb{R}_+$ and $\psi \notin \tau\left(\bar{A}\right)$: The reasoning is symmetric to the previous subcase.

4. $\{(\psi, q_\psi), (\omega, q_\omega)\} \subseteq \bar{A}$ for some $q_\psi, q_\omega \in \mathbb{R}_{\geq 0}$: It must be the case that

$$q_\psi + q_\omega \leq v_i(\{\psi, \omega\}) - v_i(\varnothing)$$

as otherwise $\bar{A}$ is not individually rational for $i$ or

$$\left\{\left(\xi, p_\xi + \epsilon \mathbb{I}_{b(\xi)=i} - \epsilon \mathbb{I}_{s(\xi)=i}\right) : \xi \in \Psi' - \tau(\bar{A})\right\}$$

is a blocking set for some small $\epsilon > 0$. In order to prevent a block by $s(\psi)$ and $j$ (using $(\hat{\psi}, q_\psi + \epsilon)$ for some small $\epsilon > 0$), we must have $q_\psi \geq t(\psi)$. Similarly, to prevent a block by $s(\omega)$ and $j$, we must have $q_\omega \geq t(\omega)$. Simple algebra shows that $t(\psi)+t(\omega) > v_i(\{\psi,\omega\})-v_i(\emptyset)$ is equivalent to the inequality $v_i(\{\psi,\omega\})+v_i(\emptyset) > v_i(\{\psi\})+v_i(\{\omega\})$, which holds as explained in the beginning of the proof of Case 1. Hence, we must have $q_\psi + q_\omega > v_i(\{\psi,\omega\}) - v_i(\emptyset)$, contradicting our earlier statement.

**Case 2** $s(\psi) = i$ and $\psi \notin \Psi$ but $\psi \in \Psi'$: Note that

$$v_i(\{\omega\}) - v_i(\varnothing) > v_i(\{\psi,\omega\}) - v_i(\{\psi\})$$

as otherwise we would have $\psi \in \Psi'$, as if

$$v_i(\{\omega\}) - v_i(\{\psi,\omega\}) \leq v_i(\varnothing) - v_i(\{\psi\})$$

then $i$ must demand to sell $\psi$ at prices $p$ if $i$ demanded to sell $\psi$ at prices $(p_{-\omega}, p'_\omega)$.

Similar to the previous case, we use the following conventions to simplify notation

$$\frac{2\left[v_i(\{\omega\}) - v_i(\{\psi,\omega\})\right] + \left[v_i(\emptyset) - v_i(\{\psi\})\right]}{3} \equiv t(\psi)$$

and

$$\frac{2\left[v_i(\{\omega\}) - v_i(\varnothing)\right] + \left[v_i(\{\psi,\omega\}) - v_i(\{\psi\})\right]}{3} \equiv t(\omega).$$

Note that due to the inequality above we must have

$$0 < t(\psi) < v_i(\{\omega\}) - v_i(\{\psi,\omega\}),$$

and

$$0 < t(\omega) < v_i(\{\omega\}) - v_i(\emptyset).$$

We have to consider two cases according to whether $s(\omega)$ is equal to $b(\psi)$ or not.

1. $s(\omega) \neq b(\psi)$: Consider the trade $\hat{\omega}$ (which must exist by exhaustivity), where $s(\hat{\omega}) = s(\omega)$ and $b(\hat{\omega}) = b(\psi) \equiv j$. Let $s(\omega)$ have preferences such that

$$u_{s(\omega)}(\Xi \cup \{\omega\}) - u_{s(\omega)}(\Xi) = u_{s(\omega)}(\Xi \cup \{\hat{\omega}\}) - u_{s(\omega)}(\Xi) = 0$$
$$u_{s(\omega)}(\Xi \cup \{\omega, \hat{\omega}\}) = -\infty$$

for all $\Xi \subseteq \Omega - \{\omega, \hat{\omega}\}$.

Now let $j$'s preferences satisfy

$$u_j(\{\hat{\omega}\} \cup \Xi) - u_j(\Xi) = t(\omega)$$
$$u_j(\{\psi\} \cup \Xi) - u_j(\Xi) = t(\psi)$$
$$u_j(\{\psi, \hat{\omega}\} \cup \Xi) - u_j(\Xi) = -\infty$$

for all $\Xi \subseteq \Omega - \{\psi, \hat{\omega}\}$.

As in the first case preferences of the above type can be extended to simple preferences over all sets of trades. We now show that no stable match can exist if preferences satisfy the above properties by distinguishing four cases.

(a) $\{\psi, \omega, \hat{\omega}\} \cap \tau(\bar{A}) = \emptyset$: In this case

$$\left\{\left(\xi, p_\xi + \epsilon \mathbb{I}_{b(\xi)=i} - \epsilon \mathbb{I}_{s(\xi)=i}\right) : \xi \in \Psi - \tau\left(\bar{A}\right) - \{\omega\}\right\} \cup \{(\omega, t(\omega))\}$$

is a blocking set as it is increases the utility of each agent except $i$ and $s(\omega)$ by at least $\epsilon$, increases the utility of $s(\omega)$ by at least $t(\omega) > 0$, and increases $i$'s utility for sufficiently small $\epsilon > 0$, since $t(\omega) < v_i(\{\omega\}) - v_i(\emptyset)$.

(b) $(\hat{\omega}, q_{\hat{\omega}}) \in \bar{A}$ for some $q_{\hat{\omega}} \in \mathbb{R}_{\geq 0}$: Given our assumptions about preferences, individual rationality (for $s(\omega)$ and $j$) requires that $\psi, \omega \notin \tau(\bar{A})$ and $q_{\hat{\omega}} \leq t(\omega)$. Since $t(\omega) < v_i(\{\omega\}) - v_i(\emptyset)$, this implies that

$$\left\{\left(\xi, p_\xi + \epsilon \mathbb{I}_{b(\xi)=i} - \epsilon \mathbb{I}_{s(\xi)=i}\right) : \xi \in \Psi - \tau\left(\bar{A}\right) - \{\omega\}\right\} \cup \{(\omega, q_{\hat{\omega}} + \epsilon)\}$$

is a blocking set and this shows that we cannot have $\hat{\omega} \in \tau(\bar{A})$.

(c) $(\omega, q_\omega) \in \bar{A}$ for some $q_\omega \in \mathbb{R}_+$ and $\psi \notin \tau(\bar{A})$: In this case $j$ obtains a utility of zero under $\bar{A}$ and in order to prevent a block by $s(\omega)$ and $j$, we must have $q_\omega \geq t(\omega)$. But then

$$\left\{\left(\xi, p_\xi + \epsilon \mathbb{I}_{b(\xi)=i} - \epsilon \mathbb{I}_{s(\xi)=i}\right) : \xi \in \Psi' - \tau\left(\bar{A}\right) - \{\psi\}\right\} \cup \{(\psi, t(\psi) - \epsilon)\}$$

is a blocking set for sufficiently small $\epsilon > 0$. To see this note first that $j$ will clearly accept all of his contracts in the blocking set, since each of these contracts increases his utility by $\epsilon > 0$. Note that $i$'s utility after the block is

$$v_i(\{\psi\}) + t(\psi) - M\epsilon,$$

where $M$ is the order of the blocking set, whereas his utility before the block is at most

$$v_i(\{\omega\}) - t(\omega).$$

Subtracting from the former the latter expression we obtain after some algebra

$$\frac{[v_i(\{\omega\}) - v_i(\varnothing)] - [v_i(\{\omega, \psi\}) - v_i(\{\psi\})]}{3} - M\epsilon,$$

which is positive for $\epsilon > 0$ sufficiently small.

(d) $\{(\psi, q_\psi), (\omega, q_\omega)\} \subseteq \bar{A}$ for some $q_\psi, q_\omega \in \mathbb{R}_{\geq 0}$: We must have that

$$q_\psi \geq v_i(\{\omega\}) - v_i(\{\psi, \omega\}),$$

since otherwise either $\bar{A}$ would not be individually rational for $i$, or

$$\left\{\left(\xi, p_\xi + \epsilon \mathbb{I}_{b(\xi)=i} - \epsilon \mathbb{I}_{s(\xi)=i}\right) : \xi \in \Psi - \tau\left(\bar{A}\right)\right\}$$

would be a blocking set. Similarly, we must have

$$q_\omega \leq v_i(\{\omega, \psi\}) - v_i(\{\psi\}).$$

We claim that $\{(\hat{\omega}, q_\omega + \epsilon)\}$ is a blocking set for $\epsilon > 0$ sufficiently small. It will clearly be chosen by $s(\omega)$, and $b(\psi)$ obtains a utility increase of at least

$$[t(\omega) - (q_\omega + \epsilon)] - [t(\psi) - q_\psi].$$

Substituting and using the price inequalities we just derived, we find that this expression is greater than or equal to

$$[v_i(\{\omega\}) - v_i(\emptyset)] - [v_i(\{\psi, \omega\}) - v_i(\{\psi\})] - \epsilon,$$

which is positive for $\epsilon$ sufficiently small.

(e) $(\psi, q_\psi) \in \bar{A}$ for some $q_\psi \in \mathbb{R}$ and $\omega \notin \tau(\bar{A})$: Then we must have

$$\begin{aligned} q_\psi &\leq t(\psi) - t(\omega) \\ &\leq v_i(\varnothing) - v_i(\{\psi, \omega\}) \end{aligned}$$

for $\{(\hat{\omega}, \epsilon)\}$ to not be a blocking set for $\bar{A}$. But then

$$Z \equiv \left\{\left(\xi, p_\xi + \epsilon \mathbb{I}_{b(\xi)=i} - \epsilon \mathbb{I}_{s(\xi)=i}\right) : \xi \in \Psi - \tau\left(\bar{A}\right) - \{\omega\}\right\} \cup \{(\omega, \epsilon)\}$$

is a blocking set for $\epsilon > 0$ sufficiently small, as $s(\omega)$ will clearly accept this contract, and $i$'s utility before is at most

$$v_i(\{\psi\}) + v_i(\varnothing) - v_i(\{\psi, \omega\})$$

and choosing from $\bar{A} \cup Z$, $i$ obtains

$$v_i(\{\omega\}) - M\epsilon$$

where $M$ is the order of the blocking set. Subtracting the former expression from the latter, we obtain

$$[v_i(\{\omega\}) - v_i(\varnothing)] - [v_i(\{\psi, \omega\}) - v_i(\{\psi\})] - M\epsilon$$

which is greater than 0 for $\epsilon > 0$ sufficiently small.

2. $s(\omega) = b(\psi) \equiv j$: Let $j$'s preferences satisfy, for all $\Xi \subseteq \Omega - \{\omega, \psi\}$,

$$\begin{aligned} u_j(\{\omega\} \cup \Xi) - u_j(\Xi) &= -t(\omega) \\ u_j(\{\psi, \omega\} \cup \Xi) - u_j(\Xi) &= t(\psi) - t(\omega) \\ u_j(\{\psi\} \cup \Xi) - u_j(\Xi) &= -\infty \end{aligned}$$

There are four subcases to consider to show that $\bar{A}$ can not be stable:

(a) $\tau(\bar{A}) \cap \{\psi, \omega\} = \varnothing$: The argument from Case 2.(a) can be used to show that $i$ and $j = s(\omega)$ have an incentive to deviate.

(b) $(\omega, q_\omega) \in \bar{A}$ for some $q_\omega \in \mathbb{R}_{\geq 0}$: Suppose that $\psi \notin \tau(\bar{A})$. Individual rationality for $j$ requires that $q_\omega \geq t(\omega)$. The argument from Case 2.(c) can then be used to establish that

$$\left\{\left(\xi, p_\xi + \epsilon \mathbb{I}_{b(\xi)=i} - \epsilon \mathbb{I}_{s(\xi)=i}\right) : \xi \in \Psi' - \tau\left(\bar{A}\right) - \{\psi\}\right\} \cup \{(\psi, t(\psi) - \epsilon)\}$$

is a blocking set for sufficiently small $\epsilon > 0$.

(c) $\{(\psi, q_\psi), (\omega, q_\omega)\} \subseteq \bar{A}$ for some $q_\psi, q_\omega \in \mathbb{R}_{\geq 0}$: We must have

$$q_\psi \geq v_i(\{\omega\}) - v_i(\{\psi, \omega\}),$$

as otherwise either $\bar{A}$ would not be individually rational for $i$, or

$$\left\{\left(\xi, p_\xi + \epsilon \mathbb{I}_{b(\xi)=i} - \epsilon \mathbb{I}_{s(\xi)=i}\right) : \xi \in \Psi - \tau\left(\bar{A}\right)\right\}$$

would be a blocking set. Similarly, we must have

$$q_\omega \leq v_i(\{\omega, \psi\}) - v_i(\{\psi\}).$$

The first inequality implies that $\bar{A}$ cannot be individually rational for $j$ since the incremental utility of signing $\psi$ on top of $\omega$ is

$$t(\psi) - q_\psi \leq t(\psi) - [v_i(\{\omega\}) - v_i(\{\psi, \omega\})] < 0.$$

(d) $\psi \in \tau(\bar{A})$ and $\omega \notin \tau(\bar{A})$: This can clearly not be individually rational for $j$, given that he obtains $-\infty$ utility if he signs $\psi$ but not $\omega$ no matter which other trades he signs.

The case where the preferences of $i$ do not satisfy part (2) of Definition 4 of substitutability is analogous. □