Optimal Advertising of Auctions*

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Abstract

We study a symmetric independent private values auction model where the revenue-maximizing seller faces a cost c_n of attracting n bidders to the auction. If the distribution of valuations possesses an increasing failure rate (IFR), the seller overinvests in attracting bidders compared to the social optimum. Conversely, if the distribution is DFR, the seller underinvests compared to the social optimum. If the distribution of valuations becomes more dispersed, both, a revenue- and a welfare-maximizing seller, attract more bidders.

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1 Introduction

We analyze a symmetric independent private values auction model where the revenue-maximizing seller faces a cost c_n of attracting n bidders. These costs can be thought of as advertising costs - or as costs of making bidders familiar with the object for sale. We mainly consider the following question: How does the number of bidders attracted by the seller compare to the socially optimal number of bidders?

Our main result is the following: If the distribution of valuations has an increasing failure rate (IFR), the seller overadvertises the auction. Conversely, if the distribution of valuations is of decreasing failure rate (DFR), the seller underadvertises. The difference in investment behavior stems from the fact that in the IFR case bidders' aggregate rents, i.e., the difference between the two highest valuations, decrease in the number of bidders. Conversely, they increase in the DFR case. Bidders' aggregate rents distinguish the revenue maximizer's and the welfare maximizer's problem. Thus they determine whether the seller over- or underadvertises.

We first develop the results for standard auctions, e.g., a second price auction without reserve price. Then we show that they generalize with minor caveats to revenue-maximizing auctions. Moreover, we show that, in an optimal auction, the seller advertises to fewer bidders than the seller in a standard auction. The reason for this is that a reserve price is more effective with fewer bidders. Thus, smaller numbers of bidders are comparatively more profitable in an optimal auction than in a standard auction without reserve. Finally, we show that, under a more dispersed distribution of valuations, both, revenue- and and welfare-maximizing sellers, increase their advertising efforts. Technically, we rely on tools from reliability theory which allow us to derive sharp results for broad classes of probability distributions.[1]

The advertising literature frequently assumes costs of attracting prospective buyers.[2] In the auction literature, such costs have received surprisingly little attention despite the fact that they are often implicitly assumed: For example, consider Bulow and Klemperer's (1996) result that attracting another bidder is more profitable than setting an optimal reserve price. Behind the comparison is the assumption that it is equally costly to set the optimal reserve or to attract one more bidder. Costs have to be involved here: If there were no costs associated with attracting more bidders,

[1]See, e.g., Barlow and Proschan (1981). For previous applications of reliability theory to the auction literature, see Li (2005) and Moldovanu, Sela and Shi (2008), and the references in these.

[2]See, e.g., Bagwell (2001).

the comparison between attracting one more bidder and implementing the optimal reserve would be unnecessary, as more bidders would be attracted anyway.

Our results are related to models where bidders strategically decide about entry to an auction such as in French and McCormick (1984), McAfee and McMillan (1987), and Levin and Smith (1994).[3] In this literature, it is assumed that the bidders (and not the seller) face costs of entering the auction. Both, McAfee and McMillan (1987), and Levin and Smith (1994), consider a revenue-maximizing seller who can influence the number of entering bidders, respectively, by entry fees or by directly restricting the pool of entrants. Both papers find that the seller's incentives coincide with social incentives. The reason is that the bidders in these models decide about entering the auction before they learn their valuations. Hence bidders end up earning zero expected profits.[4] In line with our findings, the monotonicity behavior of the bidders' aggregate rents is crucial for the relationship between revenue-maximizing and social incentives for attracting bidders: If the bidders' aggregate rents are constant (zero), the seller's interests are aligned with those of a welfare-maximizer. This coincidence of incentives stands in marked contrast to the findings in our model.

Our technical results also contribute to a recent literature which tries to develop a better understanding of Myerson's (1981) regularity condition of increasing virtual valuations.[5] We provide two results in this direction: First, we show that increasing virtual valuations are linked to monotonicity in n of the sequence of increments of expected second order statistics $E[X_{2:n+1} - X_{2:n}]$. This connection parallels the connection between the IFR condition and monotonicity of $E[X_{1:n} - X_{2:n}]$. Notably, increasing virtual valuations are in a sense a sharp condition for concavity of expected second order statistics.[6] Our proof relies on a recent observation by Ewerhart (2009): A distribution F with density f possessing increasing virtual valuations is equivalent to F possessing an increasing zoom rate

$$z_F(x) = \frac{f(x)}{(1 - F(x))^2}.$$

[3]See Bergemann and Välimäki for a recent survey which covers this literature.

[4]In McAfee and McMillan, the seller extracts all rents via ex-ante entry fees. Levin and Smith consider a symmetric equilibrium with mixing over the entry decision. There, bidders set their entry probabilities such that they all earn zero expected payoffs.

[5]See Ewerhart (2009) and the references therein.

[6]This concavity is important in our analysis since it guarantees that the seller's maximization problem is well-behaved.

Second, we show that (strictly) increasing virtual valuations are not strong enough for our purposes and for the theory of optimal auctions when considering distributions of valuations with an unbounded support: We give an example of a distribution with strictly increasing virtual valuations for which no optimal entry fee exists.

The paper proceeds as follows: Section 2 introduces the model and the optimization problems. Section 3 develops the technical tools needed for our analysis and proves concavity of first and second order statistics. Section 4 contains our results on standard auctions: We compare social and revenue-maximizing incentives for inviting bidders and then study the influence of dispersion in the distribution of valuations on the number of invited bidders. Section 5 extends our analysis to revenue-maximizing auctions. Section 6 concludes. Appendix A discusses increasing virtual valuations and related assumptions for distributions with unbounded support. All proofs are in Appendix B.

2 The Model

We consider a standard symmetric independent private values auction model with a seller who sells an indivisible object to a group of bidders. The bidders' valuations X_i are independent draws from a distribution F. We denote by $X_{k:n}$ the k^{th} largest of the random variables X_1,...,X_n and assume that $E[X_i] < \infty$.[7] There is an infinite pool of potential bidders who are initially unaware of the auction. The seller has to invest c_n to make n bidders aware of the auction. Once a bidder i becomes aware of the auction, he privately learns his valuation X_i for the object for sale. The seller values the object at zero. The object is auctioned off in a sealed-bid second price auction with reserve price r. We assume throughout that bidders adhere to their weakly dominant strategy of bidding their valuation whenever it is weakly greater than the reserve price.

We assume that the cost sequence c_n is weakly convex and strictly increasing for $n \geq 1$. Moreover, we assume $c_0 = c_1 = 0$, i.e., the seller can attract one bidder for free.[8] We assume that the distribution F possesses a continuous density f which

[7] The latter assumption ensures that all order statistics of F have finite expectation: $E[X_{k:n}] < nE[X_1] < \infty$.

[8] As will become clear below, this assumption allows us to avoid a separate discussion of the case $n = 1$. We can easily relax it to the assumption that costs grow slowly enough such that the relevant range of n is above 1.

is strictly positive over the support S of F where $S = [0, s)$ for some $s \in (0, \infty]$.[9] We assume that F fulfills Myerson (1981)'s regularity condition that the virtual valuation function V_F,

$$V_F(x) = x - \frac{1 - F(x)}{f(x)},$$

is strictly increasing in x. Moreover, we assume the following:

A1 There exists an $r^* < \infty$ such that $V_F(r^*) = 0$.

A2 The zoom rate Z_F defined by

$$Z_F(x) = \frac{f(x)}{(1 - F(x))^2}$$

converges to ∞ as x gets large.

Assumptions A1 and A2 lead to a mild but non-trivial strengthening of the increasing virtual valuations condition. They ensure that F is not only Myerson regular but that it also stays away from the borders of Myerson regularity. A1 is equivalent to assuming the existence of an optimal reserve price.[10] Since $V_F(0) = -\frac{1}{f(0)} \leq 0$, increasing virtual valuations together with A1 guarantee that virtual valuations increase strongly enough to become positive from some point on. For distributions on $\mathbb{R}^+$, increasing virtual valuations alone are not sufficient to guarantee this.[11] Assumption A2 is related to the increasing virtual valuations condition as observed by Ewerhart (2009): A distribution F has an increasing zoom rate Z_F iff V_F is increasing.[12] In this sense, A2 is - just like A1 - an assumption of sufficient growth of virtual valuations.

Our main goal is to compare the optimal choice of n under three different objectives: a) Maximizing social welfare in a second price auction with reserve price 0,[13] b) maximizing the seller's revenue in a second price auction with reserve price 0 and

[9] These regularity assumptions are made to avoid technicalities and can easily be relaxed, e.g., to densities which are zero on some intervals.

[10] See, e.g., Krishna (2002).

[11] See Appendix A for more discussion of Assumptions A1 and A2 and their relation to the increasing virtual valuations condition. There, we give an example of a distribution with strictly increasing virtual valuations for which any finite reserve price is dominated by all larger reserve prices.

[12] For a quick verification, note that both conditions correspond to the same first order condition, $2f(x)^2 + (1 - F(x))f'(x) > 0$. Ewerhart (2009) shows the equivalence under considerably weaker regularity conditions on f.

[13] Setting the reserve price to zero is obviously the welfare-maximizing choice.

c) maximizing revenue in a second price auction with reserve price chosen by the seller. The decision problem a) of a welfare-maximizing seller is given by

$$\max_n E[X_{1:n}] - c_n,$$

i.e., the seller maximizes the valuation of the winning bidder minus the invitation costs. In the following, we denote by n_w a solution to this maximization problem.

The decision problem b) of a revenue-maximizing seller who sets a reserve price of zero is given by

$$\max_n E[X_{2:n}] - c_n,$$

since the second-highest valuation is the price paid by the winning bidder. Denote by n_p a solution to this optimization problem.

Finally, the decision problem c) of a revenue-maximizing seller who sets the optimal reserve price r^* is given by

$$\max_n o_n - c_n \qquad \text{where } o_n = E\left[X_{2:n}\mathbb{1}_{\{X_{2:n}\geq r^*\}} + r^*\mathbb{1}_{\{X_{1:n}\geq r^* > X_{2:n}\}}\right],$$

since the seller's revenue is then given by r^* if only one bidder has a valuation above r^* and by the second-highest valuation if at least two bidders have a valuation above r^*. As shown by Myerson (1981) the revenue-optimal reserve price is the solution of

$$r^* = \frac{1 - F(r^*)}{f(r^*)}.$$

A solution exists by Assumption A1 and is unique by since virtual valuations are increasing. As shown by Myerson, o_n can also be written as

$$o_n = E[\max\left(V_F(X_{1:n}), 0\right)].$$

In the following sections we study the ranking of the numbers of bidders attracted to the auction, n_w, n_p and n_o, under different assumptions on F. If F is common knowledge and the bidders are aware of the revenue-maximizing seller's choice of n, maximization problem b) is equivalent to the corresponding problems for all standard auctions by the revenue-equivalence theorem. Likewise, maximization problem

c) is equivalent to the corresponding problem for all revenue-maximizing mechanisms. Common knowledge of n can arise for example if the bidders can observe n during the auction, if the seller can credibly announce n, or if the bidders can infer the seller's choice of n from his optimization problem.

Accordingly, if n is observable, the welfare-maximization problem a) is equivalent to the problem of a revenue-maximizing seller who charges entry fees before the bidders observe their valuations. The problem c) of maximizing $o_n - c_n$ is equivalent to the problem of a seller who charges an entry fee which the bidders pay after they have observed their valuations.

3 Technical Prerequisites

The following two technical observations form the basis of our analysis: First, extremal order statistics are easy to control and, second, many interesting quantities can be expressed as extremal order statistics. The next lemma establishes the first of these observations.

Lemma 1 *Let $X_1, X_2, \ldots$ be the sequence of valuations introduced above.*

(i) Let h be a weakly increasing, non-negative function with $E[h(X_1)] < \infty$ and for which $h(X_1)$ is not almost surely constant. Then $E[h(X_{1:n})]$ is a strictly increasing and strictly concave sequence. Moreover, if $\lim_{x \to s} h(x) = \infty$, then $\lim_{n\to\infty} E[h(X_{1:n})] = \infty$ where, as before, s denotes the supremum of the support of the X_i.

(ii) Let h be a weakly decreasing, non-negative function with $E[h(X_1)] < \infty$ and for which $h(X_1)$ is not almost surely constant. Then $E[h(X_{1:n})]$ is a strictly decreasing and strictly convex sequence. Moreover, if $\lim_{x \to s} h(x) = 0$, then $\lim_{n\to\infty} E[h(X_{1:n})] = 0$.

The idea behind Lemma 1 is that for an increasing function h the random variable $h(X_{1:n})$ is the first order statistic of the random variables $(Y_i)_i = (h(X_i))_i$ while for a decreasing function h the random variable $h(X_{1:n})$ is the lowest order statistic of the random variables $(Y_i)_i = (h(X_i))_i$.

From the lemma, we can immediately conclude that $E[X_{1:n}]$ and o_n are increasing and concave sequences by setting $h(x) = x$ and $h(x) = \max(V_F(x), 0)$, respectively:

Both choices of h are increasing. Our first main result shows that under increasing virtual valuations the sequence $E[X_{2:n}]$ is concave as well. Moreover, it shows that the assumption of increasing virtual valuations is, in a sense, a sharp condition for the concavity of $E[X_{2:n}]$.

Lemma 2 *It holds that*

$$E[X_{2:n+1} - X_{2:n}] = E[h(X_{1:n})]$$

where

$$h(x) = \frac{(1 - F(x))^2}{f(x)}.$$

Accordingly, under our assumptions on F, the sequence $E[X_{2:n}]$ is strictly concave and its increments go to zero as n increases.

Note that the crucial observation here is not that we can rewrite $E[X_{2:n+1} - X_{2:n}]$ in terms of $X_{1:n}$. Rather, it is the independence of n of the function h which makes the further analysis possible: Provided that h is monotone, we can now analyze $E[X_{2:n+1} - X_{2:n}]$ as a sequence of extremal order statistics. Moreover, recall that strictly increasing virtual valuations are equivalent to a strictly increasing zoom rate Z_F and accordingly equivalent to a strictly decreasing function h in Lemma 2. The strict concavity of $E[X_{2:n}]$ is thus a consequence of the increasing virtual valuations condition while the fact that the increments go to zero follows from Assumption A2.

From these concavity results we can conclude that our maximization problems are sufficiently well-behaved:

Corollary 1 *(i) If n_w maximizes $E[X_{1:n}] - c_n$, then $n_w + 2$ does not maximize $E[X_{1:n}] - c_n$.*

(ii) If n_p maximizes $E[X_{2:n}] - c_n$, then $n_p < \infty$ and $n_p + 2$ does not maximize $E[X_{2:n}] - c_n$.

The corollary shows that maximizers are almost unique[14] and that - due to the finiteness of n_p - one can always meaningfully compare n_p to n_w. In Section 5 we prove that $n_o \leq n_p$ which settles the corresponding question for optimal auctions.

[14]Due to the discrete character of the problem, uniqueness is generically fulfilled but hard to guarantee - it is easy to construct examples where two subsequent values of n are optimizers.

Finally, observe that, conversely, distributions F which exhibit strictly decreasing virtual valuations possess a strictly convex sequence $E[X_{2:n}]$ provided that $E[X_{2:n}]$ is finite.[15] This shows that the increasing virtual valuations condition is in a sense a sharp condition for concavity of second order statistics. This appears to be a novel observation.

4 Standard Auctions

In this section, we compare the numbers of bidders attracted by a revenue-maximizing seller in a standard auction, n_p, to the socially optimal number of bidders n_w. Furthermore, we show that more bidders get attracted for more dispersed distributions of valuations.

4.1 Over- and Underadvertising

The following lemma forms the basis of the comparison of the numbers of invited bidders. A revenue-maximizing seller overadvertises if his revenue $E[X_{2:n}]$ reacts more strongly to the number of bidders than welfare $E[X_{1:n}]$ and vice versa.

Lemma 3 *(i) If $E[X_{1:n}-X_{2:n}]$ is strictly increasing, it holds that $n_p \leq n_w$. Hence a revenue-maximizing seller invites less bidders than in the social optimum.*

(ii) If $E[X_{1:n}-X_{2:n}]$ is strictly decreasing, it holds that $n_p \geq n_w$. Hence a revenue-maximizing seller invites more bidders than in the social optimum.

The lemma is based on the following equivalence:

$$E[X_{1:n+1} - X_{2:n+1}] > E[X_{1:n} - X_{2:n}] \quad \Leftrightarrow \quad E[X_{1:n+1} - X_{1:n}] > E[X_{2:n+1} - X_{2:n}]$$

for all n. Thus, if $E[X_{1:n} - X_{2:n}]$ is increasing, the gains from attracting another bidder are larger with regard to social welfare than with regard to the seller's revenue.

We next identify conditions determining the monotonicity behavior of $E[X_{1:n}] - E[X_{2:n}]$. For this purpose, we apply the following result from reliability theory[16] which is also an immediate consequence of Lemma 1:

[15]For examples of such distributions see Appendix A
[16]See, e.g., Barlow and Proschan (1981).

Lemma 4 *It holds that*

$$E[X_{1:n} - X_{2:n}] = E[h(X_{1:n})] \quad where \quad h(x) = \frac{1 - F(x)}{f(x)}.$$

Accordingly, $E[X_{1:n}] - E[X_{2:n}$ *is strictly increasing if* h *is increasing and strictly decreasing if* h *is decreasing.*

The function h in the lemma is the inverse of the failure rate H_F of F which is defined by

$$H_F(x) = \frac{f(x)}{1 - F(x)}.$$

Distributions for which H_F is increasing or decreasing are known, respectively, as IFR and DFR distributions. Putting these observations together we obtain the following version of Lemma 3, which is the main result of this section:

Proposition 1 *(i) If* F *is DFR, it holds that* $n_p \leq n_w$*. Hence the seller underadvertises.*

(ii) If F *is IFR, it holds that* $n_p \geq n_w$*. Hence the seller overadvertises.*

The distinction between IFR and DFR is crucial for the tail behavior of F: The boundary case between IFR and DFR is the exponential distribution which has a constant failure rate. Here (and only here), revenue-maximizing and social incentives for attracting bidders coincide. IFR distributions form a class of distributions with lighter than exponential tails. For them, the second order statistic reacts more sensitively to changes in the number of bidders than the first order statistic. The converse holds for DFR distributions, which are more heavy-tailed than the exponential distribution.[17]

4.2 Dispersion

Let us now study how n_w and n_p react to changes in the dispersion of the distribution of the bidders' valuations. For this purpose, we stress the dependence of n_w, n_p, $E[X_{1:n}]$, and $E[X_{2:n}]$ on F by writing n_w^F, n_p^F, $E[X_{1:n}^F]$, and $E[X_{2:n}^F]$.

[17]Examples of IFR distributions are, for instance, Gaussian distributions (restricted to $\mathbb{R}^+$) and many distributions with finite support such as uniform distributions. The power-law distributions are examples of DFR distributions.

We compare the optimal advertising levels under different distributions of valuations which are ordered in the dispersive order.[18] A distribution F is said to dominate a distribution G in the dispersive order if for all $0 \leq a < b \leq 1$

$$F^{-1}(b) - F^{-1}(a) \geq G^{-1}(b) - G^{-1}(a),$$

i.e., if the distance between any pair of quantiles is larger under F than under G. This order is well-suited for our analysis since it allows to control the behavior of increments of order statistics.[19]

Expectations of order statistics - which are closely related to quantiles - lie further apart under a more dispersed distribution of valuations, see, e.g., Theorem 3.B.31 of Shaked and Shantikumar (2007). Hence both, the revenue-maximizer and the welfare-maximizer, attract more bidders the more dispersed the distribution of the bidders' valuations is:

Proposition 2 *Consider two distributions of valuations fulfilling the assumptions of our model where F dominates G in the dispersive order. Then the sequences $E[X^F_{1:n}] - E[X^G_{1:n}]$ and $E[X^F_{2:n}] - E[X^G_{2:n}]$ are increasing in n. Therefore, it holds that $n^F_w \geq n^G_w$ and $n^F_p \geq n^G_p$.*

4.3 Relaxing Assumptions A1 and A2

All results of this section still hold if we drop Assumptions A1 and A2, require only non-decreasing virtual valuations and impose additional assumptions on the cost sequence c_n to ensure finiteness of n_p. For instance, one could assume that c_n is strictly convex and that its increments become arbitrarily large as n goes to infinity. Such a generalization is not possible for the results of the next section which require the existence of a finite optimal reserve price. In Appendix A we argue that A1 and A2 are only minor strengthenings of the increasing virtual valuations assumption which are closely connected to the assumption that $E[X_i] < \infty$.

[18] See Shaked and Shantikumar (2007) for more background.

[19] Many weaker dispersion criteria such as F having a larger variance than G would not suffice for this purpose.

5 Optimal Auctions

We now come to the optimization problem of a revenue-maximizing seller who can set an optimal reserve price. Recall that we associate with this the sequence of expected revenues o_n and the optimal number of bidders n_o. Since it holds that $E[X_{2:n}] < o_n < E[X_{1:n}]$, let us consider the increments $E[X_{1:n}] - o_n$ and $o_n - E[X_{2:n}]$ in the next lemma. The monotonicity behavior of $E[X_{1:n}] - o_n$ determines whether over- or underinvestment occurs:

Lemma 5 *It holds that*

$$E[X_{1:n}] - o_n = E[h_1(X_{1:n})] \quad \textit{where} \quad h_1(x) = \min\left(x, \frac{1 - F(x)}{f(x)}\right)$$

and

$$o_n - E[X_{2:n}] = E[h_2(X_{1:n})] \quad \textit{where} \quad h_2(x) = \left(\frac{1 - F(x)}{f(x)} - x\right) \mathbb{1}_{\{x<r^*\}}.$$

By the increasing virtual valuations assumption, the function h_2 is decreasing and converges to zero. Accordingly, we obtain:

Corollary 2 *The sequence $o_n - E[X_{2:n}]$ is decreasing and converges to zero. Hence a revenue-maximizing seller who sets an optimal reserve advertises less than a revenue-maximizing seller who cannot set a reserve: $n_o \leq n_p$.*

The comparison of n_o to the welfare-optimal choice n_w is more subtle: If F is DFR, then h_1 is increasing so that we recover our underinvestment result also under optimal reserve prices:

Corollary 3 *If F is DFR, then $E[X_{1:n}] - o_n$ is increasing. Hence the revenue-maximizing seller underadvertises: $n_o \leq n_w$.*

Under IFR, the function h_1 is first increasing and then decreasing: It increases linearly until r^* and then decreases as it equals the inverse failure rate. Since increasing n moves the distribution of $X_{1:n}$ further into the right tail, the decreasing part of h_1 typically dominates for sufficiently large n. Hence, for large n, the reserve price plays a negligible role. Accordingly, we can expect to observe overinvestment under IFR distributions from some n on. We expect this effect to be more pronounced for distributions with a strongly increasing failure rate and thus a strongly decreasing inverse failure rate. We confirm this intuition with three examples: The exponential distribution which has a constant failure rate, the uniform distribution which is strongly IFR, and finally two distributions which are IFR but close to the exponential distribution.

Example 1 *If F is the exponential distribution which lies at the boundary between IFR and DFR behavior, we know that $E[X_{1:n}] - E[X_{2:n}]$ is constant. Since $o_n - E[X_{2:n}]$ decreases in n, the remainder $E[X_{1:n}] - o_n$ must increase. Thus the exponential distribution is no longer a boundary case - it behaves just like a DFR distribution. Accordingly, we observe underinvestment.*

Example 2 *If F is the uniform distribution on $[0, 1]$, and thus a distribution without tails which is "strongly" IFR, $E[X_{1:n}] - o_n$ is decreasing. Here, the decreasing part of h_1 is powerful enough to always dominate the increasing part.*[20] *Accordingly, for the uniform distribution a revenue-maximizer conducting an optimal auction overinvests.*

Example 3 *Consider the distribution F with density $f(x) = x \exp(-x)$. This distribution has the same tail behavior as the exponential distribution but it is strictly IFR. Here, the sequence $E[X_{1:n}] - o_n$ is increasing in n until $n^* = 8$ and decreases from there on: As predicted above, the IFR behavior takes over at some point. This happens despite the fact that F behaves essentially like the exponential distribution for large x. The same behavior with $n^* = 3$ is observed for Gaussian distributions restricted to $\mathbb{R}^+$. In these examples, overinvestment occurs if marginal costs are low enough to guarantee that the relevant range of n is sufficiently high.*

6 Conclusion

We have studied a symmetric independent private values auction model where the revenue-maximizing seller advertises the auction to the bidders. Our main results show that the failure rate determines whether the seller over- or underadvertises compared to the social optimum. So far, we have mainly discussed our results in the context of auction theory. We would like to conclude by discussing their place in the advertising literature.

[20] In this case, h_1 is the symmetric function $h_1 = \min(x, 1-x)$ on $[0, 1]$. Since $X_{1:n}$ has more mass on $[0.5, 1]$ than on $[0, 0.5]$ for $n > 1$, the decreasing part of h_1 is dominant for all n.

In a classical paper, Shapiro (1980) demonstrates that a revenue-maximizing monopolist who cannot price-discriminate and who can sell as many objects as he wants to underprovides informative advertising. The reason is that he cannot extract the whole surplus from the consumers: He does not fully internalize the gains from advertising more and selling to more consumers. In contrast, in our model, the product of the seller is scarce and the selling price is determined endogenously. The seller then underadvertises whenever the expected selling price in the auction reacts too little to advertising. Yet for many distribution functions, the seller overadvertises as the selling price reacts more strongly to advertising than the winning bidder's valuation and thus welfare. To our knowledge, our study is the first to capture both phenomena, over- and underadvertising, within one model.

A Discussion of Assumptions A1 and A2

In this section, we study the relation between the increasing virtual valuations assumption for distributions on $\mathbb{R}^+$, our assumption of $E[X_i] < \infty$, and Assumptions A1 and A2. Observe that conditions like increasing virtual valuations, IFR, and DFR combine two rather different types of regularity conditions: For one thing, they are strong local regularity conditions which easily get destroyed by a local perturbation of the density f. Yet in addition, they are conditions on the tail behavior of the distribution F. For the behavior of integrated quantities such as $E[X_{k:n}]$ in which we are interested, the tail behavior is more crucial.[21]

We consider two examples: First, we study power-law distributions with increasingly heavier tails and observe that increasing virtual valuations, finite expectations, A1 and A2 break down at the same heaviness of tails. This indicates that A1 and A2 do not impose additional restriction on tail behavior. Second, we provide an example of a distribution with (strictly) increasing virtual valuations which violates finite expectations, A1 and A2. This demonstrates that increasing virtual valuations alone are not strong enough both for our purposes and for the theory of optimal auctions.

[21]In contrast, local regularity is crucial when considering distributions with a finite support. Myerson's (1981) ironing technique aims at overcoming such problems of local non-regularity for finite supports.

Consider the family of power law distributions $(F_\gamma)_{\gamma>1}$ with density

$$f_\gamma(x) = (1-\gamma^{-1})(1+\gamma^{-1}x)^{-\gamma}$$

on $\mathbb{R}^+$. A larger value of γ corresponds to a distribution with lighter tails. For $\gamma > 2$, F_γ has increasing virtual valuations, has a finite expected value and satisfies A1 and A2. The distribution F_2 is a boundary case of Myerson regularity: It has constant virtual valuations and equivalently a constant zoom rate. Moreover, F_2 has infinite expectation and violates both A1 and A2: Its virtual valuations remain negative (such that the optimal reserve price is infinite) and its zoom rate is constant (such that the sequence $E[X_{2:n}]$ is linear). The fact that increasing virtual valuations, A1 and A2 break down together demonstrates that A1 and A2 are only a minor strengthening of the increasing virtual valuations condition. Moreover, we see that - viewed as an assumption on tail behavior - the increasing virtual valuations condition is the assumption that F is sufficiently light-tailed, i.e., that the density of F decays at least quadratically.[22]

To see why A1 and A2 are necessary strengthenings of the increasing virtual valuations condition consider the distribution function F given by

$$F(x) = \frac{\sqrt{1+x^2}-1}{\sqrt{1+x^2}}$$

with

$$V_F(x) = -\frac{1}{x} \quad \text{and} \quad Z_F(x) = \sqrt{\frac{x^2}{1+x^2}} < 1.$$

F possesses strictly increasing virtual valuations and a strictly increasing zoom rate. Accordingly, F possesses a strictly concave sequence of second order statistics. Yet F does not possess a finite mean and violates both A1 and A2. The sequence of second order statistics is concave but its increments are bounded away from zero. Hence for some cost sequences c_n it holds that $n_p = \infty$. Moreover, the equation $V_F(r^*) = 0$ does not possess a solution. Thus no optimal reserve price exists: In fact, any finite reserve price is dominated by all larger reserve prices in this case.

[22] The distributions F_γ are chosen such that $f_\gamma \approx x^{-\gamma}$. A positive constant is added to x to keep the density bounded in 0. Moreover, the distributions are scaled such that the limit F_∞ is the exponential distribution. Since the exponential distribution is the boundary case between IFR and DFR, the family F_γ with $\gamma > 2$ covers in a sense the tail behavior of all DFR distributions with increasing virtual valuations.

What sets this example apart from well-behaved Myerson regular distributions is that, asymptotically (for large x), it behaves like the weakly Myerson regular distribution F_2. Assumptions A1 and A2 ensure that our distributions of valuations do not get arbitrarily close to the border of Myerson regularity.[23]

An example of a strictly convex sequence $E[X_{2:n}]$ is given by the distributions F_γ with $\gamma \in (1.5, 2)$, i.e., for a non-trivial set of power law distributions which have heavier tails than those which fulfill our assumptions.[24]

B Proofs

We denote the distribution of $X_{k:n}$ by $F_{k:n}$ and its density by $f_{k:n}$. Recall that[25]

$$F_{1:n}(x) = F(x)^n, \quad f_{1:n}(x) = nF^{n-1}(x)f(x), \quad F_{2:n} = F(x)^n + nF^{n-1}(x)(1 - F(x)),$$

and $F_{n:n}(x) = 1 - (1 - F(x))^n$.

Proof of Lemma 1 Consider first independent, a.s. non-constant, positive random variables Y_i from a distribution G with finite expectation (but possibly with atoms). It holds that

$$E[Y_{1:n}] = \int_0^\infty 1 - G(x)^n dx \text{ and } E[Y_{1:n+1} - Y_{1:n}] = \int_0^\infty (1 - G(x))G(x)^n dx.$$

Since Y_1 is not almost surely constant, we have $G(x) \in (0, 1)$ on an interval of positive mass. Thus the first integral is strictly increasing and the second one is strictly decreasing in n. This shows that expectations $E[Y_{1:n}]$ of first order statistics are monotonically increasing and concave. With an analogous argument, it follows that $E[Y_{n:n}]$ is decreasing and convex. Now we consider $E[h(X_{1:n})]$. For an increasing function h, it holds that $E[h(X_{1:n})] = E[Y_{1:n}]$ where we define $Y_i = h(X_i)$. For a decreasing h, it holds that $E[h(X_{1:n})] = E[Y_{n:n}]$. This shows the monotonicity, concavity and convexity properties of $E[h(X_{1:n})]$.
Next we show that an increasing h which converges to infinity implies that $E[h(X_{1:n})]$ converges to infinity. Fix some $m > 0$. We show that, from some sufficiently large

[23] In this light, a further examination of possible dependencies between A1, A2 and the assumption of finite expectations seems to be mostly of mathematical interest since the three are closely related. This is left open for future research.
[24] For $\gamma \leq 1.5$, $E[X_{2:n}]$ becomes infinite as well.
[25] See David (1970).

n on, $E[h(X_{1:n})] > m$. By assumption, there is some $x^* \in (0, s)$ with $h(x^*) > 2m$. We can thus bound $E[h(X_{1:n})]$ by

$$E[h(X_{1:n})] = \int_0^s h(x) f_{1:n}(x) dx \geq \int_{x^*}^s 2m f_{1:n}(x) dx \geq 2m(1 - F(x^*)^n).$$

Thus we can guarantee $E[h(X_{1:n})] > m$ by choosing n sufficiently large, implying that $E[h(X_{1:n})]$ converges to infinity. The proof for a decreasing h which converges to zero is completely analogous: For sufficiently large n, most mass of $F_{1:n}$ lies on values where h is small. $\square$

Proof of Lemma 2 It holds that

$$\begin{aligned} E[X_{2:n+1} - X_{2:n}] &= \int_0^s F_{2:n}(x) - F_{2:n+1}(x) dx = \int_0^s n \left(F(x)^{n-1} - 2F(x)^n + F(x)^{n+1}\right) dx \\ &= \int_0^s nF(x)^{n-1}(1 - F(x))^2 dx = \int_0^s h(x) f_{1:n}(x) dx. \end{aligned}$$

h is decreasing by the zoom rate formulation of increasing virtual valuations and it converges to zero by A2. The results thus follow from Lemma 1. $\square$

Proof of Corollary 1 By assumption, c_n is weakly convex and, by Lemma 1, the sequences $E[X_{1:n}]$ and $E[X_{2:n}]$ are strictly concave. Thus $a_n = E[X_{1:n}] - c_n$ and $b_n = E[X_{2:n}] - c_n$ are strictly concave. If n^* and $n^* + 2$ were maximizers of a_n, we would have $a_n^* = a_{n^*+2}$ which would imply $a_{n^*+1} > \max(a_{n^*}, a_{n^*+2})$. Thus at most two subsequent numbers n can be maximizers. The same is true for b_n. Moreover, since the increments of $E[X_{2:n}]$ converge to zero by Lemma 2 and since c_n increases at least linearly, b_n decreases from some point on. Thus we must have $n_p < \infty$. $\square$

Proof of Lemma 3 If $E[X_{1:n}] - E[X_{2:n}]$ is increasing, it holds that $E[X_{1:n+1}] - E[X_{1:n}] > E[X_{2:n+1}] - E[X_{2:n}]$. Thus the value of n which balances gains and costs from attracting an additional bidder is larger under welfare-maximization than under revenue-maximization. This implies $n_w \geq n_p$. The case where $E[X_{1:n}] - E[X_{2:n}]$ is decreasing is analogous. $\square$

Proof of Lemma 4 The lemma follows from

$$E[X_{1:n} - X_{2:n}] = \int_0^s F_{2:n}(x) - F_{1:n}(x) dx = \int_0^s nF(x)^{n-1}(1 - F(x)) dx = \int_0^s h(x) f_{1:n}(x) dx$$

and from Lemma 1. $\square$

Proof of Proposition 1 The proposition is an immediate consequence of Lemma 3, Lemma 4 and the definitions of IFR and DFR. □

Proof of Proposition 2 Denote by $X_{k:n}$ and $Y_{k:n}$ the respective order statistics from F and G. From the recurrence relations on p. 45 of David (1970) and from Theorem 3.B.31 of Shaked and Shantikumar (2007), it follows that

$$E[X_{1:k} - X_{1:k-1}] = \frac{1}{k}E[X_{1:k} - X_{2:k}] \geq \frac{1}{k}E[Y_{1:k} - Y_{2:k}] = E[Y_{1:k} - Y_{1:k-1}]$$

and

$$E[X_{2:k} - X_{2:k-1}] = \frac{2}{k}E[X_{2:k} - X_{3:k}] \geq \frac{2}{k}E[Y_{2:k} - Y_{3:k}] = E[Y_{2:k} - Y_{2:k-1}].$$

Thus arguing as in the proof of Lemma 3 proves our claim. □

Proof of Lemma 5 Recall that

$$o_n = E[\max(V_F(X_{1:n}), 0)] = \int_{r^*}^{\infty} V_F(x) f_{1:n}(x) dx.$$

This implies the desired expression for $E[X_{1:n}] - o_n$. The expression for $o_n - E[X_{2:n}]$ follows from $o_n - E[X_{2:n}] = (E[X_{1:n}] - E[X_{2:n}]) - (E[X_{1:n}] - o_n)$ together with our expressions for $E[X_{1:n}] - E[X_{2:n}]$ and $E[X_{1:n}] - o_n$. □

Proof of Corollary 2 The properties of $o_n - E[X_{2:n}]$ follow directly from Lemma 1 and Lemma 5 as the function $h_2(x)$ from Lemma 5 is decreasing by the increasing virtual valuations condition and zero for $x > r^*$. Comparing the maximization problems for $E[X_{2:n}] - c_n$ and $o_n - c_n$ yields that the latter problem must have smaller solutions n_o since we add a decreasing sequence to the objective function of the former problem. □

Proof of Corollary 3 This follows from Lemma 1 and Lemma 5 as the function $h_1(x)$ from Lemma 5 is increasing if F is DFR. The inequality $n_w \geq n_o$ follows like in the proof of Lemma 3. □

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