On the optimality of small research tournaments

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Abstract

This note examines two open problems concerning the optimal number of participants in the research tournament model of Fullerton and McAfee (1999). I derive a sharp bound for the possible cost inefficiency associated with a tournament of size 2 for the case in which (asymmetric) effort costs are common knowledge and the procurer can charge non-discriminatory entry fees. The result generally supports arranging a small tournament with the two most efficient firms, with some notable exceptions. If, prior to the tournament, costs are private information of ex-ante symmetric firms, Fullerton and McAfee’s contestant selection auction has to be used to select the most efficient candidates and to raise money in advance. I discuss the procurer’s problem of stimulating a given expected aggregate research effort at lowest expected total cost by choosing the optimal tournament size. A closed form solution is derived for the case where marginal costs are uniformly distributed on [0, c]. The result strongly favors the smallest possible tournament.

1 Introduction

Research tournaments, or contests, are widely used as mechanisms to procure innovations. They may mitigate many of the problems that plague traditional procurement contracts in this case, such as non-verifiable quality of the innovation, or difficulties to monitor and verify the efforts and costs of suppliers. Properly designed contests successfully foster competition between potential suppliers, while requiring relatively

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little information on the part of the procurer.\footnote{See for instance Che and Gale (2003) and Fullerton and McAfee (1999) for very readable accounts of some of the most appealing features of contests and tournaments. These papers also provide a number of interesting examples.} In an influential paper, Fullerton and McAfee (1999) (henceforth (FM)) studied how to design a fixed-prize research tournament when firms/suppliers are heterogenous (or \textit{asymmetric}) with respect to their cost of exerting research effort, and when the research technology is \textit{stochastic}. (FM) focused on how asymmetries affect the answers to two major design questions for the procurer: how many participants should be admitted to the tournament? How should contestants be selected from a pool of \( n \) candidates when costs are \textit{private information} prior to the tournament? The purpose of this note is to address two open questions with regard to the optimal number of contestants. One problem concerns the complete information case, while the other, more fundamental one deals with the case of private information. Both problems arise naturally from the work of (FM), and my analysis builds on their model and results. I therefore summarize these before explaining the open problems and the results that I obtain.

(FM) provided a complete characterization of firms’ equilibrium research efforts and expected profits in a fixed-prize tournament with prize \( P \), when the stochastic research technology is of the “independent draws” type (leading to a standard Tullock success function, see also Baye and Hoppe, 2003), and when the heterogeneous effort costs of all \( m \) participants are common knowledge. Having more firms in the tournament increases aggregate research effort (which determines the distribution of the quality of the best innovation) but decreases equilibrium profits and hence firms’ valuations for entering the tournament. Taking this tradeoff into account, the procurer’s optimization problem involves stimulating given levels of aggregate research effort at the lowest possible total cost. For the case where the procurer can set \textit{non-discriminatory entry fees}, (FM) derived a condition on the structure of heterogeneity which ensures that it is optimal to host a very small tournament with the two most efficient firms only.\footnote{The condition is sufficient even if duplication of fixed costs plays no role.} These are selected by setting an appropriate entry fee.

Even if one has a clear idea what the optimal number of participants is, another key issue arises when effort costs are private information before the tournament: how can one make sure that the most efficient firms enter? (FM) advocated the use of an all-pay entry auction with a small interim prize (the “contestant selection auction”), in which firms bid for participation in the final tournament (with prize \( P \) and \( m \) participants). They analyzed a setup of incomplete information with ex-ante symmetric firms whose actual types, i.e. their marginal costs of conducting research, are drawn from a common and commonly known distribution. Firms’ types are private information prior to
entry. However, in case a firm is admitted, it learns about the costs of its (small number of) competitors, so that the research tournament with asymmetric firms and complete information can be used as a building block for the analysis. (FM) identified conditions on the distribution of types under which standard discriminatory-price and uniform-price entry auctions do not possess an efficient symmetric equilibrium, i.e. an equilibrium that always selects the candidates with the lowest costs. In this sense, these auctions may fail as selection mechanisms. The authors nicely demonstrated that, in sharp contrast, the contestant selection auction is much more likely to have an efficient equilibrium. In particular, this is always the case when types are independent.

I now describe the two open problems. First, while the sufficient condition for optimality of $m = 2$ in the complete information model is not implausible, there are also many interesting cases in which it is violated. In particular, this often happens when firms’ costs are drawn from a common distribution as described above. I therefore drop the condition entirely. Instead, I derive a sharp bound for the loss that may possibly occur - compared to the optimal tournament size - if entry is restricted to only two firms. I also illustrate the meaningfulness of the numerical value of this bound. The analysis confirms that setting $m = 2$ is a good idea in many cases. However, it also provides an intuitive answer as to when admitting more contestants may be particularly beneficial. Roughly speaking, this is the case if a) there is a substantial but not too extreme cost asymmetry between the two most efficient firms and b) there are other firms which are (almost) as efficient as the second best firm.

Second and importantly, it is not clear why picking a tournament size of $m = 2$ should be a good idea, or even optimal, when firms’ costs are private information. In this case, setting optimal entry fees is infeasible: the contestant selection auction has to be used to select participants efficiently and to collect money before the tournament, in the form of all-pay bids. Both (FM)’s sufficient condition and the loss analysis carried out in Section 2.2 of this note rely on complete information, and in particular on the procurer’s ability to set optimal non-discriminatory entry fees. Therefore, I further investigate the incomplete information model that was used by (FM) to argue in favor of the contestant selection auction. I discuss the expression that must be analyzed for a designer’s objective of “procuring” a given expected aggregate effort at lowest expected total cost and find a closed form solution for the important special case in which marginal costs are uniformly distributed on $[0, \bar{c}]$ (see Section 3 for additional motivation). This exercise turns out to be fairly involved but still analytically tractable (see the proof of Theorem 3). The results strongly support setting $m = 2$.

A comprehensive review of the literature on the optimal design of contests would go well beyond the scope of this note. Still, let me mention some related papers.
These also provide further references and point out connections with several important strands of the economic literature, such as the literature on labor tournaments following Lazear and Rosen (1981), and the large body of work on all-pay auctions under either complete or incomplete information. Very closely related to (FM) is a seminal contribution for the case of symmetric firms by Taylor (1995), who also found that limiting the number of contestants is beneficial. Fullerton, Linster, McKee, and Slate (2002) and Schöttner (2008) compared fixed-prize tournaments to first-price auctions (firms bid a combination of quality and price after having conducted research) in two different models with stochastic research technology and symmetric firms. Che and Gale (2003) studied a model with deterministic technology, complete information and (possibly) asymmetric firms. They showed that inviting the two most efficient firms to participate in a first-price auction (handicapping the better firm by a maximum allowable price) is optimal within a much broader class of possible contests. In a model with a given number of ex-ante symmetric participants, deterministic technology and privately known cost types, Moldovanu and Sela (2001) demonstrated that it may be better to award multiple prizes rather than just a single one if the cost function is convex. However, a single prize is better for concave or linear costs (as is the case here).

The note is organized as follows. Section 2.1 describes (FM)’s complete information tournament model with asymmetric firms, introduces notation, and collects the known results that are needed later on. Section 2.2 studies the complete information model without the sufficient condition of (FM). In particular, the sharp bound mentioned above is developed. Section 3 contains the analysis and results for the incomplete information model, in which the contestant selection auction is used to determine participants. All proofs that are not in the main text may be found in Section 4.

2 Optimal tournament size under complete information

2.1 The model and some known results

In this section, I present the basic complete information tournament model, as well as the results from (FM) that will be used.\textsuperscript{3} I adopt their notation so as to facilitate a comparative reading.

Risk-neutral firms $j = 1, \ldots, n$ ($n \geq 2$) have costs for making an effort $z_j \in \mathbb{R}_+$

\textsuperscript{3}For more details, their paper should be consulted.
when they take part in a simultaneous-move research tournament with \( m \) participants and prize \( P > 0 \). Costs are given by \( \gamma + c_j z_j \) (where \( \gamma \geq 0 \) and \( c_j > 0 \)) if \( z_j > 0 \), and by 0 if \( z_j = 0 \).\(^4\) If firm \( j \) exerts effort \( z_j \), it produces an innovation of random quality \( x_j \in [0, \bar{x}] \), whose c.d.f. is given as \( F^{z_j}(x_j) \). This simple stochastic research technology corresponds to a continuous version of modeling independent draws from the distribution \( F \), which is assumed to be absolutely continuous with respect to Lebesgue measure. Firms’ research activities are independent, so that \( F^Z \) is the distribution function of the winning quality, where \( Z = \sum_j z_j \) is the aggregate effort. In this model, winning probabilities are determined by a standard Tullock success function. That is, the probability that firm \( j \) produces the highest quality and wins the tournament (and hence the prize \( P \)) is \( \frac{z_j}{Z} \). Each others’ costs are common knowledge among the \( m \) participants.

(FM) showed the following: if \( M \) denotes the set of contestants (\( |M| \leq m \)) who choose strictly positive effort in equilibrium, then for each \( i \in M \), effort \( z_i \) and expected profit \( \pi_i \) before subtracting \( \gamma \) are given by

\[
\begin{align*}
  z_i &= \frac{P(|M| - 1)}{\sum_{j \in M} c_j} \left[ 1 - \frac{c_i(|M| - 1)}{\sum_{j \in M} c_j} \right], \\
  \pi_i &= P \left[ 1 - \frac{c_i(|M| - 1)}{\sum_{j \in M} c_j} \right]^2.
\end{align*}
\]

If \( \gamma > 0 \), the additional constraint \( \pi_i \geq \gamma \) must be satisfied for all \( i \in M \). (FM) showed that there is a unique set of lowest-cost contestants who choose strictly positive effort in equilibrium if \( \gamma = 0 \) (which is the case that I consider below). There may be some ambiguity if \( \gamma > 0 \). However, even in that case one may essentially restrict attention to the equilibrium in which only (a \( \gamma \)-dependent number of) lowest-cost contestants actively participate.\(^5\) For convenience, and following (FM), I label all \( n \) firms such that marginal costs are ordered: \( c_1 \leq c_2 \leq ... \leq c_n \).

It is intuitive, and confirmed by the model, that high fixed costs are a strong reason for having a small number of participants. To enhance a clear analysis of the consequences of asymmetries in marginal costs for the optimal size of the tournament, I eliminate this effect and set \( \gamma = 0 \). Note that

\[
Z = \sum_{i \in M} z_i = \frac{P(|M| - 1)}{\sum_{j \in M} c_j},
\]

\(^4\)The fixed cost \( \gamma \geq 0 \) is incurred only if \( z_j > 0 \).
\(^5\)See Lemma 1 in (FM).
and that it is profitable for firm $m$ to be in a tournament with $1, \ldots, m - 1$ if and only if

$$
\frac{(m - 1)c_m}{\sum_{j=1}^{m} c_j} < 1.
$$

(2)

This is condition (5) in (FM). It is straightforward to see that the left hand side of (2) is strictly increasing in $m$ as long as it is smaller than 1, and that if $2 \leq \tilde{m} \leq n$ is the maximal number such that (2) is satisfied for all $m \leq \tilde{m}$, then the condition is violated for all $k$ with $n \geq k > \tilde{m}$ (compare (FM)).

For a given $P$, varying the number of participants from 2 up to $\tilde{m}$ goes along with increasing total effort. Indeed, let $m + 1 \leq \tilde{m}$, so that (2) implies $c_{m+1} < \frac{\sum_{j=1}^{m+1} c_j}{m}$ which is also equivalent to $c_{m+1} < \frac{\sum_{j=1}^{m} c_j}{m-1}$. For the comparison of total efforts with $m + 1$ and $m$ contestants this yields

$$
\frac{Pm}{\sum_{j=1}^{m+1} c_j} > \frac{Pm}{\frac{m}{m-1} \sum_{j=1}^{m} c_j} = \frac{P(m-1)}{\sum_{j=1}^{m} c_j}.
$$

On the other hand, firms’ equilibrium profits decrease with more participants, which diminishes the procurer’s ability to collect money in advance, through entry fees or entry auctions, that may be used to partly finance the prize.

As (FM) noted, with complete information the procurer can charge a non-discriminatory entry fee of (slightly below)

$$
E = P \left[ 1 - \frac{c_m(m-1)}{\sum_{j=1}^{m} c_j} \right]^2
$$

to induce a tournament with exactly $m \leq \tilde{m}$ low-cost participants. If he wants to motivate a target effort of $Z$, by (1) he must set $P = \frac{Z \sum_{j=1}^{m} c_j}{m-1}$. Hence, the total cost of “procuring” effort $Z$ with $m \leq \tilde{m}$ competitors is given by

$$
TC_m = P - mE = \frac{Z \sum_{j=1}^{m} c_j}{m-1} \left( 1 - m \left[ 1 - \frac{c_m(m-1)}{\sum_{j=1}^{m} c_j} \right]^2 \right) = Z \sum_{j=1}^{m} c_j \left( -1 + 2\Delta_m - \frac{m-1}{m} \Delta_m^2 \right),
$$

(3)

where $\Delta_m = \frac{mc_m}{\sum_{j=1}^{m} c_j}$. In the symmetric case where all costs are equal to the same $c$, all $TC_m$ are equal to $Zc$ (just plug in $\Delta_m = 1$). So, the interesting questions in the non-discriminatory complete information framework are the following. How do asymmetries in marginal costs (which imply that some firms earn net profits) affect
the $TC_m$? Do asymmetries generally favor any particular number of participants, do they at least tend to do so? A theorem of (FM) partly answers these questions.

**Theorem 1.** *(Theorem 3 of (FM))* If $\Delta_m$ is nondecreasing in $m$, then the total cost $TC_m$ of stimulating a given aggregate effort level $Z$ is minimized at $m = 2$.

### 2.2 A sharp bound for general asymmetric cost structures

The sufficient condition of Theorem 1 requires that marginal costs are separated from each other by certain gaps. It has some appeal, but there are also many cases of interest in which it does not hold. For instance, the condition does not apply whenever one of the firms is a sufficiently close competitor for another one. Most importantly, the condition is violated for many realizations when firms’ costs are randomly drawn from an ex-ante distribution (see Section 3). I do not make any assumptions about the structure of marginal costs (reflected by the ratios $\Delta_m$) here. Consequently, $m = 2$ is not always optimal. However, it is possible to derive a sharp, worst case, lower bound for the ratio of the optimal $TC_m$ over $TC_2$. This bound is quite close to 1 while, in contrast, having too many firms in the tournament can be very expensive for some cost structures (see Example 1 for an illustration). In this sense, hosting a tournament with the two most efficient firms is generally not a bad idea. However, the analysis also yields the following insights as by-products. First, it is better to have more than two contestants whenever there exist close competitors for firm 2. Secondly, increasing the number of participants can be really profitable only for intermediate degrees of asymmetry between firms 1 and 2, and the effect is maximal for $\Delta_2 = \sqrt{2}$ (see Remark 1).

**Theorem 2.** It holds

$$\inf_{n \geq 2, 0 < c_1 \leq \ldots \leq c_n, 1 \leq m < n} \frac{TC_{m+1}}{TC_2} = \inf_{\Delta_2 \in [1, 2)} \frac{1}{2} \left[ 2 \left( 1 - \sqrt{\frac{1}{2}} \right) - \left( \sqrt{\frac{1}{2} \Delta_2 - \sqrt{\frac{1}{2}}} \right)^2 \right]$$

$$= \frac{1}{4 \left( 1 - \sqrt{\frac{1}{2}} \right)} \approx 0.85.$$

(4)

To prove Theorem 2, I first rewrite formula (3) in a form that is more suitable for

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6The argument is reinforced by the observations that the bound is developed for the case $\gamma = 0$, and that there may be important additional costs related to conducting the tournament (evaluation, etc.) that lie beyond the framework of the model.
examining the ratios $\frac{TC_{m+1}}{TC_2}$, $\bar{m} > m \geq 1$.

\[
TC_{m+1} = Z \sum_{j=1}^{m+1} c_j \left( -1 + 2\Delta_{m+1} - \frac{m}{m+1} \Delta_{m+1}^2 \right)
\]
\[
= Z \frac{(m+1)c_{m+1}}{\Delta_{m+1}} \left[ 2 \left( 1 - \sqrt{\frac{m}{m+1}} \right) \Delta_{m+1} - \left( \sqrt{\frac{m}{m+1}} \Delta_{m+1} - \sqrt{\frac{1}{\Delta_{m+1}}} \right)^2 \right]
\]
\[
= Z(m+1)c_{m+1} \left[ 2 \left( 1 - \sqrt{\frac{m}{m+1}} \right) - \left( \sqrt{\frac{m}{m+1}} \Delta_{m+1} - \sqrt{\frac{1}{\Delta_{m+1}}} \right)^2 \right]
\]

This yields for $\bar{m} > m \geq 1$,

\[
\frac{TC_{m+1}}{TC_2} = \frac{(m+1)c_{m+1}}{2c_2} \left[ 2 \left( 1 - \sqrt{\frac{m}{m+1}} \right) - \left( \sqrt{\frac{m}{m+1}} \Delta_{m+1} - \sqrt{\frac{1}{\Delta_{m+1}}} \right)^2 \right]
\]
\[
= \left( \prod_{j=2}^{m} \alpha_j \right) \frac{(m+1)}{2} \left[ 2 \left( 1 - \sqrt{\frac{m}{m+1}} \right) - \left( \sqrt{\frac{m}{m+1}} \Delta_{m+1} - \sqrt{\frac{1}{\Delta_{m+1}}} \right)^2 \right]
\]

(5)

Here $\alpha_j := \frac{c_{j+1}}{c_j}$, and an empty product is equal to 1 by convention. By the ordering of costs, $\Delta_{m+1} \geq 1$. On the other hand, $\frac{m}{m+1} \Delta_{m+1} < 1$ is necessary for firm $m+1$ to enter and to exert positive effort (i.e., for $m < \bar{m}$). The following lemma establishes a straightforward recursive formula for $\Delta_m$, as well as a necessary and sufficient condition for the implication ($m \leq \bar{m} \Rightarrow m+1 \leq \bar{m}$) in terms of the sequence of marginal costs.

**Lemma 1.**  i) It holds

\[
\Delta_{m+1} = (m+1)c_{m+1} \Delta_m.
\]

ii) If $\frac{m-1}{m} \Delta_m < 1$, then

\[
\frac{m}{m+1} \Delta_{m+1} < 1 \iff c_{m+1} < \left( \frac{m-1}{m} \Delta_m \right)^{-1} c_m.
\]

Consequently, each structure of marginal costs that may lead to a situation in which $m+1$ firms participate in equilibrium is characterized by a sequence of $\alpha_j$ satisfying

\[
\alpha_j \in \left[ 1, \left( \frac{j-1}{j} \Delta_j \right)^{-1} \right) \text{ for } j = 2, \ldots, m.
\]
This is all information that is available a priori for studying how small the expression (5) may get for arbitrary sequences of marginal costs (that do not satisfy an additional monotonicity property like in Theorem 1). At first glance, it thus seems rather difficult to find a sharp lower bound for the ratio (5). The next observation drastically simplifies this task.

**Lemma 2.** Consider any $2 \leq m \leq \bar{m}$. If it holds in addition that $c_{m+1} = c_m$, then $TC_{m+1} \leq TC_m$, i.e. it is weakly better to have a tournament with $m+1$ rather than $m$ participants. The inequality is strict unless $\Delta_m = 1$.

Lemma 2 enables the following proof of Theorem 2.

**Proof of Theorem 2.** Lemma 2 shows that for any situation in which a tournament with $m > 2$ firms is better than one with 2, one may construct a case in which total costs are even lower with $m+1$ firms (by adding another firm whose marginal cost equals that of firm $m$).

Returning to the expression for the ratio of total costs (5), $\frac{m}{m+1} \Delta_{m+1} \in \left[ \frac{m}{m+1}, 1 \right]$ and $\Delta_{m+1}^{-1} \in (\frac{m}{m+1}, 1]$ imply

$$\left( \sqrt{\frac{m}{m+1} \Delta_{m+1}} - \sqrt{\frac{1}{\Delta_{m+1}}} \right)^2 = O\left( \frac{1}{m^2} \right).$$

This follows from $\sqrt{x} = 1 + \frac{1}{2}(x-1) + O((x-1)^2)$ as $x \to 1$, which furthermore yields

$$1 - \sqrt{\frac{m}{m+1}} = \frac{1}{2(m+1)} + O\left( \frac{1}{m^2} \right).$$

Hence, asymptotically only the latter term determines the numerator in (5). For a given $\Delta_2 \in [1, 2)$, one may thus reach convergence to the infimum of (5) over all admissible cost structures by setting $\alpha_j = 1$ for all $j \geq 2$. The infimum is given by

$$\lim_{m \to \infty} \frac{2(m+1) \left( 1 - \sqrt{\frac{m}{m+1}} \right)}{2 \left[ 2 \left( 1 - \frac{1}{2} \right) - \left( \sqrt{\frac{1}{2} \Delta_2} - \sqrt{\frac{1}{\Delta_2}} \right)^2 \right]} = \frac{1}{2 \left[ 2 \left( 1 - \sqrt{\frac{1}{2}} \right) - \left( \sqrt{\frac{1}{2} \Delta_2} - \sqrt{\frac{1}{\Delta_2}} \right)^2 \right]}.$$

This expression is minimized at $\Delta_2 = \sqrt{2}$ with value $\frac{1}{4(1-\sqrt{2})}$. □

**Remark 1.** As indicated above, the proof yields some additional insights: whether it can be really profitable to allow entry of more than two firms depends crucially on the asymmetry between the two strongest firms. In the extremal cases of equal strength ($\Delta_2 = 1$) and drastic superiority of firm 1 ($\Delta_2 \to 2$), $m = 2$ is always optimal. Indeed,
in these cases the denominator of the ratio (5) is equal to 1. The closer the asymmetry is to the geometric mean of these extremes, the more profitable it may be to allow for more contestants to increase competition, but only if there are other firms which are (almost) as strong as firm 2.

To conclude this section, I construct an ad hoc example which shows that the cost inefficiency from having too many firms in the tournament may be significantly higher than the one associated with the worst case bound of Theorem 2.

**Example 1.** Consider \( \Delta_j = 1 + \frac{j-2}{m-1} \frac{1}{m} \) for \( j = 2, ..., m + 1 \), and for \( m = 5 \). Then \( \frac{T_6}{T_2} \approx 1.38 \).

### 3 Incomplete information, (FM)’s contestant selection auction, and optimal tournament size

In this section, I study the optimal number of participants for the model of *incomplete information* with ex-ante symmetric firms that was used by (FM) to argue in favor of the contestant selection auction. The crucial observation, which seems to have gone unnoticed in (FM), is that a separate and more involved analysis is needed for that purpose. Both Theorem 1 and the results of Section 2.2 have been obtained assuming complete information and the use of non-discriminatory entry fees. Therefore, these results are not applicable (even though they are useful to guide intuition).

The type of each firm, i.e. its marginal cost, is drawn ex-ante from a common and commonly known distribution \( H \) with support \([c, \bar{c}] \subset \mathbb{R}_+\) and density \( h \). Types are private information prior to the tournament. However, they become common knowledge among the selected contestants before effort choices are made, so that efforts and profits can be computed as in Section 2.1. (FM) showed that the contestant selection auction, an all-pay entry auction with a small interim prize for entry, performs well in selecting the most efficient contestants under incomplete information. In contrast, entry fees, uniform-price auctions and discriminatory-price auctions may perform very poorly (see also Section 1). In particular, for independent types the contestant selection auction always has a symmetric equilibrium with a bidding function that is strictly decreasing in marginal cost (Theorem 5 of (FM)). Moreover, for every \( P \) and \( m \), the expected total cost of conducting the tournament, which takes the revenue from the pre-tournament auction into account, is then formally equivalent to the expression that would arise for an efficient uniform-price auction (Theorem 6 of (FM)). To be

\[ \Delta_{m+1} \]

For a fully rigorous argument, \( \Delta_{m+1} \) has to be slightly below \( \frac{m+1}{m} \).
precise about the latter point, let \( h_{n-m,n} \) denote the density of the \( m+1 \)st - lowest out of \( n \) independent draws from \( H \). Moreover, let \( \psi(c,c) \) denote the conditional expected profit of a firm of type \( c \) which enters the tournament as the weakest one of \( m \) contestants, and which happens to have the same cost as the \( m \)th-strongest out of all remaining \( n-1 \) candidates (the “marginal” firm).\(^8\) Lemma 3 below follows easily from Theorem 6 of (FM).\(^9\)

**Lemma 3.** With independent types, the expected total cost of conducting a tournament with prize \( P \), when the \( m \) participants are determined by means of (FM)’s contestant selection auction, is

\[
P - m \int \psi(c,c) h_{n-m,n}(c) \, dc. \tag{6}
\]

For the rest of this note, it will be convenient to drop the assumption that \( c_1,\ldots,c_{m-1} \) are ordered. Spelling out \( \psi(c,c) \) explicitly for \( \gamma = 0 \), one finds (compare (FM), where the “max” was forgotten):

\[
\psi(c,c) = \int_c^c \ldots \int_c^c P \max \left( 1 - \frac{c(m-1)}{c + \sum_{j=1}^{m-1} c_j}, 0 \right)^2 \prod_{j=1}^{m-1} \frac{h(c_j)}{H(c_j)} \, dc_1 \ldots dc_{m-1}. \tag{7}
\]

Absent further (and necessarily special) assumptions about the procurer’s utility as a function of i) the quality of the winning innovation, and ii) monetary costs, the most natural and interesting optimization problem is the following, which is analogous to the one in the complete information case: “procure” a given expected effort \( \bar{Z} \) at the lowest possible expected total cost. Remember from (1) that the relationship between prize \( P \), total equilibrium effort \( Z = \sum_{j \in M} z_j \) (where \( M \) is the set of firms that make strictly positive effort) and marginal costs is \( Z = \frac{P(|M|-1)}{\sum_{j \in M} c_j} \). Thus, to target an expected effort of \( \bar{Z} \) with \( m \) participants, the procurer must set

\[
P = \frac{\bar{Z}}{E_n \left[ \sum_{|M|, M \subseteq \{1,\ldots,n\}} \frac{|M|-1}{\sum_{j \in M} c_j} \, |M| \leq m \right]}.
\]

The conditional expectation in the denominator is the expected total effort made by the \( m \) lowest cost firms (which are selected by the all-pay auction) in a tournament with prize 1. The subscript \( n \) indicates the dependence of this expression on the total number of firms. Plugging \( \psi(c,c) \) and the relationship for \( P \) into (6), one obtains an

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\(^8\)\( \psi(c,c) \) is the usual candidate for equilibrium bidding in a uniform-price auction.

\(^9\)For completeness, a proof is included in Section 4. The reason is that (FM) presented their results in a slightly different model, in which firms/agents are characterized by an ability parameter \( w \). \( w \) and \( c \) are related by \( w = \frac{1}{c} \).
expression for the expected total cost of procuring $\tilde{Z}$ from $m$ contestants.

$$ETC_{m,n}^2 := \frac{\tilde{Z}}{E_n \left[ \frac{|M|-1}{\sum_{j \in M} c_j} \right] |M| \leq m} \times$$

$$\int_{c}^{\bar{c}} \int_{[c,c]}^{m-1} \left[ 1 - m \max \left( 1 - \frac{c(m-1)}{c + \sum_{j=1}^{m-1} c_j}, 0 \right) \right]^2 \prod_{j=1}^{m-1} \frac{h(c_j)}{H(c)} dc_1...dc_{m-1} h_{n-m,n}(c) dc.$$

(8)

While this looks similar to the first line of (3), formula (8) is not just an average of the $TC_m$. Rather, the expectations of the cost of conducting a tournament with prize $P = 1$ and of the corresponding total effort are taken separately.

It seems very hard to find general conditions that relate properties of $H$ to the optimal $m$, for each given $n$. In the remainder of this note, I solve the important special case $H \sim U[0,\bar{c}]$ explicitly (without further loss of generality, $\bar{c} = 1$). The results show that the smallest possible tournament, i.e. $m = 2$, is by far the most cost efficient one. Let me give some motivation first. $\bar{c} = 0$ is important to make the problem interesting. Indeed, the assumptions of incomplete information prior to the tournament and complete information after entry seem reasonable only if $n$ is quite large. On the other hand, if $\bar{c} > 0$, then for given $H$ and large $n$, there are no significant asymmetries between the lowest-cost firms. Without fixed costs, it then does not really matter whether e.g. $m = 2$ or $m = 3$: if $\bar{c} > 0$, it holds that

$$\lim_{n \to \infty} ETC_{m,n}^2 \left[ 1 - m \left( 1 - \frac{m-1}{m} \right)^2 \right] = \tilde{Z}\bar{c}.$$

In contrast, $\bar{c} = 0$ may induce an interesting problem where asymmetries between the few best candidates remain important even though the total number of firms is large. The case $H \sim U[0,1]$ has nice additional homogeneity properties. In particular, certain conditional expectations of expressions that depend on marginal costs relative to each other only, such as $\frac{c(m-1)}{c + \sum_{j=1}^{m-1} c_j}$, do not depend on $n$. This is very helpful for computations.

In Theorem 3 below, I do the main step towards solving the optimization problem by deriving a closed form solution for $ETC_{m,n}^2$.

**Theorem 3.** Let $H \sim U[0,1]$. Then for any $n \geq 2$ and $2 \leq m \leq n$, the expected total
Table 1: Expected money raised per contestant for $P = 1$, normalized aggregate expected effort for $P = 1$, and normalized expected total cost, for $m = 2, ..., 6$.

<table>
<thead>
<tr>
<th>$m$</th>
<th>$\kappa_m$</th>
<th>$\ln 2 + \sum_{j=3}^{m} \frac{\beta_j}{(j-1)!}$</th>
<th>$\frac{n}{Z} ETC_{m,n}^{Z}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>0.113706</td>
<td>0.693147</td>
<td>1.11461</td>
</tr>
<tr>
<td>3</td>
<td>0.012558</td>
<td>0.716395</td>
<td>1.34329</td>
</tr>
<tr>
<td>4</td>
<td>0.001373</td>
<td>0.718123</td>
<td>1.38487</td>
</tr>
<tr>
<td>5</td>
<td>0.000142</td>
<td>0.718268</td>
<td>1.39125</td>
</tr>
<tr>
<td>6</td>
<td>0.000014</td>
<td>0.718281</td>
<td>1.39210</td>
</tr>
</tbody>
</table>

Theorem 3 settles the question about the optimal number of contestants for the case $H \sim U[0, 1]$. It reveals that two contestants are optimal because the decrease in expected money raised by the entry auction that goes along with increasing tournament size strongly dominates the positive effect of increasing aggregate effort. This is shown in Table 1 for $m = 2, ..., 6$. Note in particular the big jump in expected costs from $m = 2$ to $m = 3$. 

The cost for stimulating expected effort $\bar{Z}$ in a tournament with $m$ participants is

$$ETC_{m,n}^{Z} = \frac{\bar{Z}}{n} \frac{1 - m\kappa_m}{\ln 2 + \sum_{j=3}^{m} \frac{\beta_j}{(j-1)!}},$$

where $\kappa_2 = \frac{3}{2} - 2\ln 2$, and for $m \geq 3$

$$\kappa_m = \frac{1}{(m-2)!} \left[ (\ln(m-1) - \ln m) \left( (m-2)m^{m-3} - 2(m-1)m^{m-2} + m^m \right) \right]$$
$$+ \frac{1}{(m-2)!} \left[ \sum_{j=0}^{m-4} \left( \frac{m-2}{m-3-j} - \frac{2(m-1)}{m-2-j} + \frac{m}{m-1-j} \right) m^j \right]$$
$$+ \frac{1}{(m-2)!} \left[ \left( 2 - \frac{3}{2} m \right) m^{m-3} + m^{m-1} \right],$$

and

$$\beta_m = (m-1) \left[ -m^{m-2}(\ln(m-1) - \ln m) - \sum_{j=0}^{m-3} \frac{m^j}{m-2-j} \right]$$
$$+ (m-2) \left[ (m-1)^{m-2}(\ln(m-2) - \ln(m-1)) + \sum_{j=0}^{m-3} \frac{(m-1)^j}{m-2-j} \right].$$

Theorem 3 settles the question about the optimal number of contestants for the case $H \sim U[0, 1]$. It reveals that two contestants are optimal because the decrease in expected money raised by the entry auction that goes along with increasing tournament size strongly dominates the positive effect of increasing aggregate effort. This is shown in Table 1 for $m = 2, ..., 6$. Note in particular the big jump in expected costs from $m = 2$ to $m = 3$. 

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4 Appendix

Proof of Lemma 1. To establish i), note that
\[
\frac{1}{\Delta_{m+1}} = \frac{\sum_{j=1}^{m+1} c_j}{(m+1)c_{m+1}} = \frac{1}{\Delta_m} \frac{mc_m}{(m+1)c_{m+1}} + \frac{1}{m+1} = \frac{mc_m + \Delta_m c_{m+1}}{(m+1)c_{m+1}\Delta_m}.
\]

ii) By i), we have
\[
\frac{m}{m+1} \Delta_{m+1} < 1 \iff mc_{m+1} \Delta_m < mc_m + \Delta_m c_{m+1} \iff (m-1)c_{m+1} \Delta_m < mc_m
\]
\[
\iff c_{m+1} < \frac{m}{m-1} \Delta_m^{-1} c_m.
\]

\[\square\]

Proof of Lemma 2. From the technical appendix of (FM) it follows that
\[
TC_{m+1} - TC_m = Z \sum_{j=1}^{m+1} c_j \left[ (\Delta_{m+1} - \Delta_m) \left( 2 \frac{m}{m+1} \Delta_{m+1} - \frac{m-1}{m} \Delta_m \right) \right]
\]
\[
+ Z \sum_{j=1}^{m+1} c_j \left[ \frac{\Delta_{m+1}}{m+1} (\Delta_m - 1) \left( 1 - \frac{m-1}{m} \Delta_m \right) \right].
\]

The factor multiplied with \(Z \sum_{j=1}^{m+1} c_j\) has been split into two additive parts merely due to limitations of space. I examine this factor for the case \(c_{m+1} = c_m\), which by
Lemma 1 implies $\Delta_{m+1} = \frac{(m+1)\Delta_m}{m+\Delta_m}$. This yields

$$(\Delta_{m+1} - \Delta_m) \left( 2 - \frac{m}{m+1} \Delta_{m+1} - \frac{m-1}{m} \Delta_m \right) + \frac{\Delta_{m+1}}{m+1} (\Delta_m - 1) \left( 1 - \frac{m-1}{m} \Delta_m \right)
$$

$$= \left( \frac{(m+1)\Delta_m}{m+\Delta_m} - \Delta_m \right) \left( 2 - \frac{m\Delta_m}{m+\Delta_m} - \frac{m-1}{m} \Delta_m \right)
$$

$$+ \frac{\Delta_m}{m+\Delta_m} (\Delta_m - 1) \left( 1 - \frac{m-1}{m} \Delta_m \right)
$$

$$= \Delta_m (1 - \Delta_m) \left( \frac{2(m+\Delta_m) - m\Delta_m - \frac{m-1}{m} \Delta_m (m+\Delta_m)}{m+\Delta_m} \right)
$$

$$+ \frac{(\Delta_m - 1) (m-1) \Delta_m^2}{m+\Delta_m}
$$

$$= \frac{\Delta_m - 1}{(m+\Delta_m)^2} \left[ (m+\Delta_m) \left( \Delta_m - m \Delta_m^2 \right) - \Delta_m \left( 2m + (3-2m) \Delta_m - \frac{m-1}{m} \Delta_m^2 \right) \right]
$$

$$= \frac{\Delta_m - 1}{(m+\Delta_m)^2} \left[ m\Delta_m + \Delta_m^2 - (m-1)\Delta_m^2 - 2m\Delta_m - (3-2m)\Delta_m^2 \right]
$$

$$= \frac{\Delta_m - 1}{(m+\Delta_m)^2} \left[ (m-1)\Delta_m^2 - m\Delta_m \right]
$$

$$= \frac{\Delta_m(\Delta_m - 1)}{(m+\Delta_m)^2} \left[ (m-1)\Delta_m - m \right]
$$

Note that $(m-1)\Delta_m - m < 0$ by assumption. The other factor is $\geq 0$, and equality holds only in the boundary case $\Delta_m = 1$. This proves the claim. \hfill \Box

Calculations for Example 1. Note that $\Delta_2 = 1$ and $\Delta_{m+1} = \frac{m+1}{m}$. (5) implies

$$\frac{TC_6}{TC_2} = \left( \prod_{j=2}^{5} \alpha_j \right) \frac{6 \left( 1 - \sqrt{\frac{5}{6}} \right) \left( 1 + \sqrt{\frac{5}{6}} \right)}{2 \left( 1 - \sqrt{\frac{1}{2}} \right) \left( 1 + \sqrt{\frac{1}{2}} \right)} = \prod_{j=2}^{5} \alpha_j.
$$

From Lemma 1,

$$\Delta_{j+1} = \frac{(j+1)c_{j+1}}{jc_j + \Delta_j c_{j+1}} \Delta_j = \frac{\Delta_j}{\Delta_j/j + \Delta_{j+1}},
$$

which yields after a few steps

$$\alpha_j = \frac{j \Delta_{j+1}}{\Delta_j (j+1 - \Delta_{j+1})}.
$$

For the current example, this implies $\alpha_2 = \frac{2(1+\sqrt{2})}{3(1+\sqrt{2})} = \frac{14}{13}$, Similarly, $\alpha_3 = \frac{220}{203}, \alpha_4 =$

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\( \frac{920}{847}, \alpha_5 = \frac{25}{23} \) and thus

\[
\frac{TC_6}{TC_2} = \prod_{j=2}^{5} \alpha_j \approx 1.38.
\]

\( \square \)

**Proof of Lemma 3.** In (FM), \([w, \bar{w}]\) (or \([w, +\infty)\)) denotes the support of the distribution of an ability type, with c.d.f. \(\tilde{H}\) and density \(\tilde{h}\). \(\tilde{h}_{m+1,n}\) is the density of the \(m+1\)-1\st\ highest draw out of \(n\) independent draws from \(\tilde{H}\), and \(\tilde{\psi}(w, w)\) is the analog of \(\psi(c, c)\). According to Theorem 6 of (FM) then, with independent types, the expected total cost of conducting a tournament with \(m\) contestants when these are selected by the contestant selection auction is

\[
P - m \int_{\bar{w}}^{\bar{w}} \tilde{\psi}(w, w) \tilde{h}_{m+1,n}(w) dw.
\]

Set \(\bar{w} := \frac{1}{c}, \bar{w} := \frac{1}{c}\), and \(w = \frac{1}{c}\) in between. Define \(\tilde{\psi}(w, w)\) via \(\tilde{\psi}(w, w) := \psi(\frac{1}{\bar{w}}, \frac{1}{\bar{w}})\).

Since \(dc = \frac{-1}{w^2} dw\), and since increasing \(c\) means decreasing \(w\), it holds that \(\tilde{h}(w) = -h(c) \frac{dc}{dw} = \frac{1}{w^2} h(\frac{1}{w})\). Moreover, \(\tilde{H}(w) = 1 - H(\frac{1}{w})\).

From the general formula for order statistics, it follows that

\[
\tilde{h}_{m+1,n}(w) = \frac{n!}{(n-m-1)! m!} \tilde{H}(w)^{n-m-1} \left(1 - \tilde{H}(w)\right)^m \tilde{h}(w)
\]

\[
= \frac{n!}{m! (n-m-1)!} \tilde{H} \left(\frac{1}{w}\right)^m \left(1 - \tilde{H} \left(\frac{1}{w}\right)\right)^{n-m-1} \frac{h \left(\frac{1}{w}\right)}{w^2}
\]

Hence, expression (9) becomes

\[
P - m \int_{\bar{w}}^{\bar{w}} \tilde{\psi} \left(\frac{1}{w}, \frac{1}{w}\right) \frac{h_{n-m,n} \left(\frac{1}{w}\right)}{w^2} dw = P - m \int_{\bar{w}}^{\bar{w}} \psi(c, c) h_{n-m,n}(c) dc.
\]

\( \square \)

**Proof of Theorem 3.** Observe that \(\frac{h(c)}{H(c)} = \frac{1}{c}\). Let \(\text{vol}_k(A)\) denote the \(k\)-dimensional Hausdorff measure of set \(A\). Making use of the Coarea formula for the mapping \((c_1, \ldots c_{m-1}) \mapsto \sum_{j=1}^{m-1} c_j\), I find for the crucial term in the second factor of \(ETC_{m,n}^{Z}\) in
(8), for \( m \geq 3 \):

\[
\begin{align*}
\int_{[0,c]^{m-1}} \max \left( 1 - \frac{c(m-1)}{c + \sum_{j=1}^{m-1} c_j}, 0 \right)^2 \frac{1}{c^{m-1}} dc_1 \ldots dc_{m-1} \\
= \frac{1}{c^{m-1}} \int_{(m-2)c}^{(m-1)c} \left( 1 - \frac{(m-1)c}{c + a} \right)^2 \text{vol}_{m-2} \left( \left\{ \sum_{j=1}^{m-1} c_j = a \right\} \cap [0,c]^{m-1} \right) \sqrt{\frac{1}{m-1}} da \\
= \frac{1}{c^{m-1}} \int_0^c \left( 1 - \frac{(m-1)c}{mc - a} \right)^2 \text{vol}_{m-2} \left( \left\{ \sum_{j=1}^{m-1} c_j = (m-1)c - a \right\} \cap [0,c]^{m-1} \right) \sqrt{\frac{1}{m-1}} da \\
= \frac{1}{c^{m-1}} \int_0^c \left( \frac{c - a}{mc - a} \right)^2 \text{vol}_{m-2} \left( \left\{ \sum_{j=1}^{m-1} c_j = a \right\} \cap [0,c]^{m-1} \right) \sqrt{\frac{1}{m-1}} da \\
= \frac{1}{c^{m-1}} \int_0^c \left( \frac{c - a}{mc - a} \right)^2 \frac{\sqrt{m-1}}{(m-2)!} \sqrt{\frac{1}{2m-2}} \sqrt{\frac{1}{m-1}} da \\
= \frac{1}{c^{m-1}(m-2)!} \int_0^c \left( \frac{c^2a^{m-2} - 2ca^{m-1} + a^m}{(mc - a)^2} \right) da. 
\end{align*}
\]

Expression (10) shows that a formula for integrals of the form \( \int_0^c \frac{a^k}{(b-a)^2} da \) for \( k \geq 1 \) and \( b > c \) is needed. Integrating by parts once,

\[
\int_0^c \frac{a^k}{(b-a)^2} da = \frac{c^k}{b-c} - a \int_0^c \frac{a^{k-1}}{b-a} da.
\]

Noting that \( a^{k-1} = a^{k-1} - b^{k-1} + b^{k-1} = (a-b) \left( \sum_{j=0}^{k-2} a^{k-2-j} b^j \right) + b^{k-1} \) (with the convention that sums running from \( j = 0 \) to \(-1\) are zero), it is straightforward to establish

\[
\int_0^c \frac{a^k}{(b-a)^2} da = \frac{c^k}{b-c} + k b^{k-1} \left( \ln(b-c) - \ln b \right) + \sum_{j=0}^{k-2} k b^j \frac{c^{k-1-j}}{k-1-j}.
\]

After plugging in and collecting terms, which I omit here for the sake of brevity,
equation (10) turns into:

\[
\int_{[0,c]^{m-1}} \max \left(1 - \frac{c(m-1)}{c + \sum_{j=1}^{m-1} c_j}, 0\right)^2 \frac{1}{c^{m-1}} dc_1 \ldots dc_{m-1} = \frac{1}{(m-2)!} \frac{1}{m^{m-1}} [\ln(m-1) - \ln m] (m-2) m^{m-3} - 2(m-1) m^{m-2} + m^m]
\]

\[
+ \frac{1}{(m-2)!} \sum_{j=0}^{m-4} \left( \frac{m-2}{m-3-j} - \frac{2(m-1)}{m-2-j} + \frac{m}{m-1-j} \right) m^j
\]

\[
+ \frac{1}{(m-2)!} \left( 2 - \frac{3}{2} \right) m^{m-3} + m^{m-1} = \kappa_m.
\]

This expression does not depend on \(c\) or \(n\), so that \(\kappa_m\) is also the expected money raised per contestant in the entry auction (for \(P = 1\)), irrespectively of \(n\). A separate calculation is necessary for \(\kappa_2\).

\[
\kappa_2 = \int_0^c \left(1 - \frac{c}{c + c_1}\right)^2 \frac{1}{c} dc_1 = \frac{1}{c^2} \int_0^c \left(\frac{c_1}{c + c_1}\right)^2 dc_1
\]

\[
= \frac{1}{c} \left[ -\frac{c}{2} + \int_0^c \frac{2c_1}{c + c_1} dc_1 \right] = -\frac{1}{2} + \frac{2}{c} \left[ c \ln(2c) - \int_0^c \ln(c + c_1) dc_1 \right]
\]

\[
= -\frac{1}{2} + \frac{2}{c} \left[ c \ln(2c) - 2c \ln(2c) + c + c \ln c \right]
\]

\[
= \frac{3}{2} + 2(\ln c - \ln(2c)) = \frac{3}{2} - 2 \ln 2.
\]

Concerning the terms \(\gamma_{m,n} := E_n \left[ \frac{|M| - 1}{\sum_{j \in M} c_j} |M| \leq m \right]\), a separate calculation is again needed for \(m = 2\).

\[
\gamma_{2,n} = E_n \left[ \frac{|M| - 1}{\sum_{j \in M} c_j} |M| \leq 2 \right] = \int_0^1 \int_0^c \frac{1}{c + c_1} \frac{1}{c} dc_1 n(n-1)(1-c)^{n-2} dc
\]

\[
= (\ln 2) n(n-1) \int_0^1 (1-c)^{n-2} dc = n \ln 2.
\]

While it is practically impossible to compute the terms for higher \(m\) directly, they may be computed recursively. Indeed, allowing for an additional contestant has an effect on aggregate equilibrium effort only in those cases where the cost of the new firm is low enough in the sense of inequality (2). Also, in all these relevant cases, if the new firm is not allowed to participate, then all the lower cost firms make positive effort because of the monotonicity of the right hand side of (2) that was mentioned
in Section 2.1. I use the equivalent formulation of (2) which was also mentioned in Section 2.1 to describe the domain of integration. Consequently, for \( m \geq 3, \)

\[
\gamma_{m,n} - \gamma_{m-1,n} = \int_0^1 \frac{h_{n-m+1,n}(c)}{c^{m-1}} \int_{[0,c]^{m-1}} \left( \frac{m-1}{c + \sum_{j=1}^{m-1} c_j} - \frac{m-2}{\sum_{j=1}^{m-1} c_j} \right) I_{\{(m-2)c < \sum_{j=1}^{m-1} c_j \}} \, dc_1 \cdots dc_{m-1} \, dc.
\]

The inner integral may again be computed by using the Coarea formula and the little symmetry trick:

\[
\int_{[0,c]^{m-1}} \left( \frac{m-1}{c + \sum_{j=1}^{m-1} c_j} - \frac{m-2}{\sum_{j=1}^{m-1} c_j} \right) I_{\{(m-2)c < \sum_{j=1}^{m-1} c_j \}} \, dc_1 \cdots dc_{m-1}
\]

\[
= \int_{(m-2)c}^{(m-1)c} \left( \frac{m-1}{c + a} - \frac{m-2}{a} \right) \operatorname{vol}_{m-2} \left( \left\{ \sum_{j=1}^{m-1} c_j = a \right\} \cap [0,c]^{m-1} \right) \sqrt{\frac{1}{m-1}} \, da
\]

\[
= \int_0^c \left( \frac{m-1}{mc - a} - \frac{m-2}{(m-1)c - a} \right) \operatorname{vol}_{m-2} \left( \left\{ \sum_{j=1}^{m-1} c_j = a \right\} \cap [0,c]^{m-1} \right) \sqrt{\frac{1}{m-1}} \, da
\]

\[
= \int_0^c \left( \frac{m-1}{mc - a} - \frac{m-2}{(m-1)c - a} \right) 1 \sqrt{m-1} a^{m-2} \sqrt{\frac{1}{m-1}} \, da
\]

\[
= \frac{1}{(m-2)!} \int_0^c \left( (m-1) \frac{a^{m-2}}{mc - a} - (m-2) \frac{a^{m-2}}{(m-1)c - a} \right) \, da.
\]

Now,

\[
\int_0^c \left( (m-1) \frac{a^{m-2}}{mc - a} - (m-2) \frac{a^{m-2}}{(m-1)c - a} \right) \, da
\]

\[
= (m-1) \left[ -(mc)^{m-2}(\ln(m-1) - \ln m) - \sum_{j=0}^{m-3} \frac{m^j}{m-2-j} c^{m-2} \right]
\]

\[
+ (m-2) \left[ ((m-1)c)^{m-2}(\ln(m-2) - \ln(m-1)) + \sum_{j=0}^{m-3} \frac{(m-1)^j}{m-2-j} c^{m-2} \right]
\]

\[
= c^{m-2} \beta_m.
\]
Hence,

\[
\gamma_{m,n} - \gamma_{m-1,n} = \int_0^1 \frac{h_{n-m+1,n}(c)}{e^{n-1}} \frac{1}{(m-2)!} c^{m-2} \beta_m dc \\
= \beta_m \frac{1}{(m-2)!} \int_0^1 n! \frac{1}{(n-m)! (m-1)!} (1-c)^{n-m} c^{m-2} dc \\
= \beta_m \frac{1}{n} \frac{n!}{(m-2)! (n-m)! (m-1)!} \frac{(m-2)! (n-m)!}{(n-1)!} \\
= \frac{n}{(m-1)!} \beta_m.
\]

This shows

\[
\gamma_{m,n} = n \left( \ln 2 + \sum_{j=3}^{m} \frac{\beta_j}{(j-1)!} \right).
\]

Q.E.D.

References


