Abstract

We study a two-sided matching model with a finite number of agents who are characterized by privately known, multi-dimensional attributes that jointly determine the match surplus of each potential partnership. Utility functions are quasi-linear, and monetary transfers among agents are feasible. We ask the following question: what divisions of surplus within matched pairs are compatible with information revelation leading to the formation of an efficient (surplus-maximizing) matching? Our main result shows that the only robust rules compatible with efficient matching are those that divide realized surplus in a fixed proportion, independently of the attributes of the pair’s members. In other words, to enable efficient match formation, it is necessary that each agent expects to get the same fixed percentage of surplus in every conceivable match. A more permissive result is obtained for one-dimensional attributes and supermodular value functions.

Keywords: Matching, Sharing Rule, Multi-dimensional Attributes

1 Introduction

We study a two-sided one-to-one matching (or assignment) market with a finite number of privately informed agents that need to be matched to form productive relationships. For ease of reference, we call the two sides of the market “workers” and
“employers”. Agents are characterized by multi-dimensional attributes which determine the match value / surplus potentially created by each employer-worker pair. Attributes are private information, and our model is an incomplete information version of the famous assignment game due to Shapley and Shubik (1971). In particular, for each pair, match surplus is informationally interdependent, a natural feature in the context of matching models.

We consider situations in which the ex-post realized match surplus in every partnership is divided according to some standardized contract or “sharing rule”. For instance, such a sharing rule might result from specifying claims to various components of joint surplus (or fixed shares of these). If partners’ attributes are ex-post verifiable, the rule might also determine shares directly as a function of these attributes. The implied division of surplus determines agents’ utilities in every possible matching, net of additional monetary transfers that are decided upon when workers and employers compete for partners under incomplete information, i.e. in the match formation phase. We ask the following question: can we characterize standardized divisions of surplus that are compatible with information revelation leading, for each realization of attributes in the economy, to an efficient matching? Our main result shows that in settings with multi-dimensional, complementary attributes, the only sharing rules that may induce efficient match formation are those that divide the surplus in each match according to a fixed proportion, independently of the attributes of the pair’s members. Thus, to enable efficient matching it is necessary and sufficient that all workers expect to get the same fixed percentage of surplus in every conceivable match, and the same thing must hold for employers! More flexibility is possible when attributes are one-dimensional and match surplus is supermodular. Efficient matching is then compatible with any division that leaves each partner with a fraction of the surplus that is also supermodular.

The equilibrium notion used here is the \textit{ex-post equilibrium}. This is a generalization of equilibrium in dominant strategies appropriate for settings with interdependent values, and it embodies a notion of no regret: chosen actions must be considered optimal even after the private information of others is revealed. Ex-post equilibrium is a belief-free notion, and our results do not depend in any way on the distribution of attributes in the population.\footnote{See also Bergemann and Morris (2005) for the tight connections between ex-post equilibria and “robust design”.
}
An interesting illustration for a fixed-proportion rule is offered by the German law governing the sharing of profit among a public sector employer and an employee arising from the employee’s invention activity. The law differentiates between universities and all other public institutions.\(^2\) Outside universities - where, presumably, the probability of an employee making a job-related discovery is either nil or very low - the law allows any \textit{ex-ante} negotiated contract governing profit sharing (see §40-1 in \textit{Bundesgesetzblatt III, 422-1}). In marked contrast, independently of circumstances, any university and any researcher working there must divide the profit from the researcher’s invention according to a \textit{fixed 30%-70% rule}, with the employee getting the 30% share (see §42-4). The rigidity of this “no-exception” rule is additionally underlined by an explicit mention that all feasible arrangements under §40-1 are not applicable within universities (see §42-5).

The occurrence of inflexible, fixed-proportion rules for sharing ex-post surplus - shares do not vary with attributes and are not the object of negotiation - is a recurrent theme in several interesting literatures that try to explain this somewhat puzzling phenomenon. For example, Newbery and Stiglitz (1979) and Allen (1985), among many others, noted that sharecropping contracts in many rural economies involve shares of around one half for landlord and tenant. This percentage division is observed in widely differing circumstances and has persisted in many places for a considerable length of time.\(^3\) Another example is the \textit{commenda}, a rudimentary form of company that formed for the duration of a single shipping mission in medieval Venice (for more details and impact on Venice’s extraordinary economic expansion see Acemoglu and Robinson, 2012). A commenda involved two partners, one who stayed at home, and one who accompanied the cargo. Only two types of contracts were possible: either the sedentary partner provided 100% of the capital and received 75% of the profits, or he provided 67% of the capital and received 50% of the profits.

Our study is at the intersection of several important strands of the economic literature. We briefly review below some related papers from each of these strands, emphasizing both the existing relations to our work and the present novel aspects.

1. **Matching:** An overwhelming majority of studies within the large body of work on two-sided markets has assumed either complete information or \textit{private values}

\(^2\)Practically all German universities are public.

\(^3\)For example, Chao (1983) noted that a fixed 50-50 ratio was prevalent in China for more than 2000 years. The French and Italian words for “sharecropping” literally mean “50-50 split”.
models, that is, models where agents’ preferences only depend on signals available to them at the time of the decision, but not on signals privately available to others. This holds for both the Gale-Shapley (1962) type of models with ordinal preferences, and for the branch studying variants and applications (to auctions and double auctions, say) of the assignment model with quasi-linear preferences due to Shapley and Shubik (1971).

In the Gale-Shapley model, one-sided serial dictatorship where women, say, sequentially choose partners according to their preferences leads to a Pareto-optimal matching. Difficulties occur when the stronger stability requirement is invoked: a standard result is that no stable matching can be implemented in dominant strategies if both sides of the market are privately informed (see Roth and Sotomayor, 1990). Chakraborty, Citanna and Ostrovsky (2010) showed that stability may fail even in a one-sided private information model if preferences on one side of the market (colleges, say) depend on information available to agents on the same side of the market.

Becker (1973) has popularized a special case of the complete information Shapley-Shubik model where agents have one-dimensional attributes, and where the match value is a supermodular function of these attributes. In particular, agents are completely ordered according to their marginal productivity, and efficient matching is assortative. If two-sided incomplete information is introduced in the Becker model one immediately obtains interdependent values, i.e., agents’ preferences also depend on signals available to others. But, somewhat surprisingly, there are only very few such models: most of the literature has assumed either complete information, or one-sided private information (which yields private values), or a continuum of types (so that aggregate uncertainty disappears). An exception is Hoppe, Moldovanu and Sela (2009) who analyzed a two-sided matching model with a finite number of privately informed agents, characterized by complementary one-dimensional attributes. In their model match surplus is divided in a fixed proportion, and they showed that efficient, assortative matching can arise as one of the equilibria of a bilateral signaling game. This finding is consistent with the results of the present paper.

Another strand using Becker’s specification and complete information has combined matching with ex-ante investment: before matching, agents undertake costly investments that affect their attributes and hence, ultimately, their match values. In two recent studies in this vein, Mailath, Postlewaite and Samuelson (2012a, 2012b)
focused on the role of what they call “premuneration values”, i.e., the surplus accruing to agents from matching, net of additional monetary transfers. They detailed how these values are affected by the specification of property rights.\footnote{They also noted that in certain circumstances it may be difficult to adjust premuneration values due to legal restrictions, prevailing social norms, or non-contractible components of match value.} Under personalized pricing - that must finely depend on the attributes of the matched pairs - an equilibrium which entails efficient investment and matching always exists in large (continuum) markets, no matter how surplus is shared.\footnote{Thus, as in Cole, Mailath and Postlewaite (2001), market competition eliminates hold-up problems.} In contrast, when personalized pricing is not feasible, premuneration values affect both the incentives to invest and final payoffs, and under-investment typically occurs.

2. Property Rights: Our study is also related to the large literature analyzing the effects of the ex-ante allocation of property rights on bargaining outcomes, following Coase (1960). Traditionally, this literature has not placed the bargaining agents in an explicit market context. The interplay between private information and ex-ante property rights in private value settings was emphasized by Myerson-Satterthwaite (1983) and Cramton-Gibbons-Klemperer (1987) in a buyer-seller framework and a partnership dissolution model, respectively. Fieseler, Kittsteiner and Moldovanu (2003) offered a unified treatment that allows for interdependent values and encompasses both the above private values models and Akerlof’s (1970) market for lemons. In all these papers, agents have one-dimensional types and a value maximizing allocation can be implemented via standard Clarke-Groves-Vickrey schemes. Whenever inefficiencies for certain allocations of property rights occur, these stem from the inability to design budget-balanced and individually rational transfers that sustain the value maximizing allocation.\footnote{With several buyers and sellers, the Myerson-Satterthwaite model becomes a one-dimensional, linear incomplete information version of the Shapley-Shubik assignment game. Only in the limit, when the market gets very large, one can reconcile, via almost efficient double-auctions, incentives for information revelation with budget-balancedness and individual rationality.} Brusco, Lopomo, Robinson and Viswanathan (2007) and Gärtner and Schmutzler (2009) looked at mergers with interdependent values, a setting which is more related to the present study.\footnote{However, at most one match is formed in these models, and private information consists of, or can be reduced to, one-dimensional types.} They focused on the difficulties that privately known stand-alone values pose for designing combinations of property rights and budget-balanced and individually rational transfers that lead to value maximizing mergers.
In marked contrast to all the above papers, our present analysis completely abstracts from budget-balancedness and individual rationality. In our setting, stand-alone values are known and forming a match between two agents is always better than leaving them as singles, but who is matched to whom is crucial for allocative efficiency. The fixed-proportion sharing rules are dictated here by the mere requirement of value maximization together with incentive compatibility.

3. Multi-dimensional Attributes and Mechanism Design: The prevalent assumption that agents can be described by a single trait such as skill, technology, wealth, or education is often not tenable. Workers, for example, have very diverse job-relevant characteristics, which are only partially correlated. Tinbergen (1956) pioneered the analysis of labor markets where jobs and workers are described by several characteristics. The study of complete information assignment models with a continuum of traders and multi-dimensional attributes has been pioneered by Gretsky, Ostroy and Zame (1992, 1999). Dizdar (2012) generalized the matching cum ex-ante investment model due to Cole, Mailath and Postlewaite (2001) along this line.\(^8\)

The present combination of multi-dimensional attributes, private information and interdependent values is usually detrimental to efficient implementation. In fact, Jehiel, Meyer-ter-Vehn, Moldovanu and Zame (2006) have shown that, generically, only trivial social choice functions - where the outcome does not depend on the agents’ private information - can be ex-post implemented when values are interdependent and types are multi-dimensional. Jehiel and Moldovanu (2001) have shown that, generically, the efficient allocation cannot be implemented even if the weaker Bayes-Nash equilibrium concept is used.

Our present insight can be reconciled with those general negative results by noting that the two-sided matching model is not generic. In particular, we assume here that match surplus has the same functional form for all pairs (as a function of the respective attributes), and that the match surplus of any pair depends neither on how agents outside that pair match, nor on what their attributes are. These features are natural for the matching model but are “non-generic”. Moreover, fixed-proportion sharing can be interpreted as using some limited amount of ex-post information (for each pair, the surplus realized by that pair) to determine final payoffs, and it is known that such features may potentially aid implementation (see Mezzetti, 2004 and Remark 1

\(^8\)Like other recent, related literature (e.g. Chiappori, McCann and Nesheim, 2010) his analysis used tools borrowed from optimal transportation theory. See Villani (2009) for an excellent textbook.
The sufficiency of fixed-proportion sharing for incentive compatibility is related to the presence of individual utilities that admit a cardinal alignment with social welfare: aggregate surplus becomes a *cardinal potential*, as defined by Jehiel, Meyer-ter-Vehn and Moldovanu (2008).\(^9\) By proving necessity of fixed-proportion rules,\(^10\) we identify a class of interesting settings for which efficient implementation is possible only if social welfare is a cardinal potential. Our result is also reminiscent of Roberts’ (1979) characterization of dominant strategy implementation in private values settings, but both technical assumptions and proof are very different here. The analysis of the special case with one-dimensional types and supermodular match surplus is based on an elegant characterization result due to Bergemann and Välimäki (2002), who generalized previous insights due to Jehiel and Moldovanu (2001) and Dasgupta and Maskin (2000).

Finally, in a recent contribution, Che, Kim and Kojima (2012) have shown that efficiency is not compatible with incentive compatibility in a one-sided assignment model where agents’ values over objects are allowed to depend on information of other agents. Their signals are one-dimensional but inefficiency occurs there because of the assumed lack of monetary transfers.

The paper is organized as follows: in Section 2 we present the matching model. In Section 3 we state our results, both for the multi-dimensional case and for the special case of one-dimensional attributes and supermodular surplus. Section 4 concludes. All proofs are in the Appendix.

## 2 The Matching Model

There are \(I\) employers and \(J\) workers. Each employer \(e_i\) (\(i \in \mathcal{I} = \{1, \ldots, I\}\)) privately knows his type \(x_i \in X\), and each worker \(w_j\) (\(j \in \mathcal{J} = \{1, \ldots, J\}\)) privately knows his type \(y_j \in Y\). The supports of agents’ possible types, \(X\) and \(Y\), are open connected subsets of Euclidean space \(\mathbb{R}^n\) for some \(n \in \mathbb{N}\). If an employer of type \(x\) and a worker of type \(y\) form a match, they subsequently create a *match surplus* of \(v(x, y)\), where \(v : X \times Y \to \mathbb{R}_+\) is continuously differentiable. Unmatched agents create zero surplus,

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\(^9\)They presented several non-generic cases where ex-post implementation is possible. See also Bikchandani (2006) for other such cases, e.g. certain auction settings.

\(^10\)There is some minor additional flexibility if the rule is not required to be independent of whether employers or workers are on the long side of the market, see Theorem 1.
and all agents have quasi-linear utilities.

The set of alternatives $\mathcal{M}$ consists of all possible one-to-one matchings of employers and workers. If $I \leq J$, these are the injective maps $m : I \to J$. A matching $m \in \mathcal{M}$ will be called efficient for a type profile $(x_1, \ldots, x_I, y_1, \ldots, y_J)$ if and only if it maximizes aggregate surplus $u_m(x_1, \ldots, x_I, y_1, \ldots, y_J) = \sum_{i=1}^{I} v(x_i, y_{m(i)})$ among all $m' \in \mathcal{M}$. Analogous definitions apply for the case $J \leq I$. Efficient matchings can be obtained as the solutions of a finite linear program (Shapley and Shubik, 1971).

2.1 Sharing Rules

A sharing rule specifies a standardized division of match surplus. As noted in the introduction, such a division might result from defining claims to (fixed shares of) different components of joint surplus (whose relative contribution to full match surplus may depend on attributes), or, if partners’ attributes are ex-post verifiable, shares may be defined as a function of both attributes. We introduce the following notation: if $e_i$ and $w_j$ are matched in $m \in \mathcal{M}$, then employer $e_i$’s share of surplus is $v_{e_i}^m(x_1, \ldots, x_I, y_1, \ldots, y_J) = \gamma(x_i, y_j)v(x_i, y_j)$, and worker $w_j$’s share is $v_{w_j}^m(x_1, \ldots, x_I, y_1, \ldots, y_J) = (1 - \gamma(x_i, y_j))v(x_i, y_j)$, where we assume that $\gamma : X \times Y \to \mathbb{R}$ is continuously differentiable.\(^{11}\) If $e_i$ remains unmatched in $m$ we have $v_{e_i}^m(x_1, \ldots, x_I, y_1, \ldots, y_J) = 0$ (similarly, $v_{w_j}^m(x_1, \ldots, x_I, y_1, \ldots, y_J) = 0$ if $w_j$ stays unmatched). Note that, for each realization of attributes, the set of value-maximizing matchings does not depend on $\gamma$. It will follow easily that, for efficient matching to be implementable, it is necessary that $\gamma$ is $[0, 1]$-valued (to provide strict incentives for truth-telling, it is necessary that $\gamma$ is $(0, 1)$-valued).

Together with a sharing rule, the previously described matching model gives rise to a natural social choice setting with interdependent values. Every agent attaches a value to each possible alternative, i.e. to matchings of employers and workers. This value depends both on the agent’s own type and on the type of the partner, but not on the private information of other agents. Moreover, this value does not depend on how other agents match. Thus, there are no allocative externalities, and there are no informational externalities across matched pairs.

\(^{11}\)In principle, one might allow that shares depend on the identities of the partners. Given that match surplus is determined by productive attributes only, this would not be conducive to efficient implementation. This intuition may easily be formalized using the techniques from the Appendix.
2.2 Mechanisms

By the Revelation Principle, we may restrict attention to direct revelation mechanisms where truthful reporting by all agents forms an ex-post equilibrium. A direct revelation mechanism (mechanism hereafter) is given by functions \( \Psi : X^I \times Y^J \rightarrow \mathcal{M} \), \( t^e : X^I \times Y^J \rightarrow \mathbb{R} \) and \( t^w : X^I \times Y^J \rightarrow \mathbb{R} \), for all \( i \in I \), \( j \in J \). \( \Psi \) selects a feasible matching as a function of reports, \( t^e \) is the monetary transfer to employer \( e_i \), and \( t^w \) is the monetary transfer to worker \( w_j \), as functions of reports.

Truth-telling is an ex-post equilibrium if for all employers \( e_i \), for all workers \( w_j \), and for all type profiles \( p = (x_1, ..., x_I, y_1, ..., y_J) \), \( p' = (x_1, ..., x_i', ..., x_I, y_1, ..., y_J) \) and \( p'' = (x_1, ..., x_I, y_1, ..., y_j', ..., y_J) \), it holds that

\[
\begin{align*}
    v^e_{\Psi(p)}(p) + t^e(p) & \geq v^e_{\Psi(p')}^e(p') + t^e(p') \\
    v^w_{\Psi(p)}(p) + t^w(p) & \geq v^w_{\Psi(p'')}^w(p'')+ t^w(p'').
\end{align*}
\]

3 The Main Results

We now turn to our main question: which sharing rules, if any, are compatible with information revelation leading to an efficient matching? In other words, using the mechanism design terminology, we ask for which induced utility functions can we implement the value-maximizing social choice function in ex-post equilibrium.

For our main results we need an assumption known as the twist condition in the mathematical literature on optimal transport (see Villani, 2009). This is a multidimensional generalization of the well-known Spence-Mirrlees condition. While in optimal transport - where measures of agents are matched - the condition is invoked in order to ensure that the optimal transport, corresponding here to the efficient matching, is unique and deterministic, we use it for quite different, technical reasons (see the lemmas in the Appendix).

**Condition 1**  
\( i ) \) For all \( x \in X \), the continuous mapping from \( Y \) to \( \mathbb{R}^n \) given by \( y \mapsto (\nabla_X v)(x, y) \) is injective.

\( ii ) \) For all \( y \in Y \), the continuous mapping from \( X \) to \( \mathbb{R}^n \) given by \( x \mapsto (\nabla_Y v)(x, y) \) is injective.
Match surplus functions that fulfill Condition 1 model many interesting complementarities between multi-dimensional types of workers and employers. In particular, \( v \) is not additively separable with respect to \( x \) and \( y \), so that the precise allocation of match partners really matters for efficiency.\(^{12}\) As a simple example consider the bilinear match surplus: \( v(x, y) = x \cdot y \), where \( \cdot \) denotes the standard inner product on \( \mathbb{R}^n \). Then \((\nabla_X v)(x, y) = y\) and \((\nabla_Y v)(x, y) = x\), and Condition 1 is satisfied.

We can now state the central results concerning the necessity and sufficiency of fixed-proportion sharing:

**Theorem 1** Let \( n \geq 2, I, J \geq 2 \), and assume that Condition 1 is satisfied. Then the following are equivalent:

i) The efficient matching is implementable in ex-post equilibrium.

ii) There is a constant \( \lambda_0 \in [0,1] \) and functions \( g : X \to \mathbb{R} \) and \( h : Y \to \mathbb{R} \) such that for all \( x \in X, y \in Y \) it holds that

\[
(\gamma v)(x, y) = \lambda_0 v(x, y) + g(x) + h(y),
\]

and, moreover, \( h \) is constant if \( I < J \), and \( g \) is constant if \( I > J \).

**Corollary 1** The only sharing rules that can implement the efficient matching irrespective of whether employers or workers are on the short side of the market are of the form \( (\gamma v)(x, y) = \lambda_0 v(x, y) + c \), where \( \lambda_0 \in [0,1] \) and \( c \) is a constant.

It is easy to show that under the conditions of Theorem 1 ii), it is possible to align all agents’ utilities with aggregate surplus (via appropriate Clark-Groves-Vickrey type transfers). When the part of the share that is proportional to match surplus is strictly positive for both sides of the market (i.e. \( \lambda_0 \in (0,1) \)), then a *strict* cardinal alignment is possible: in this case, aggregate surplus is a *cardinal potential* for the individual utilities (Jehiel, Meyer-ter-Vehn and Moldovanu, 2008).

Proving that Theorem 1 ii) is necessary for efficient implementation is much more difficult. The heart of our proof is concerned with situations with two agents on each side, and it exploits the implications of incentive compatibility on the part of employers for varying worker type profiles. Condition 1 ensures that the subset of types for

\(^{12}\)Note for instance that if \( v \) is additively separable and \( I = J \), then all matchings are efficient, and hence the efficient matching can trivially be implemented, no matter what \( \gamma \) is. This stands in sharp contrast to the result of Theorem 1.
which both feasible matchings are efficient is a well-behaved manifold. As mentioned earlier, our result is reminiscent of Roberts’ (1979) Theorem that shows (under some relatively strong technical conditions) that any dominant-strategy implementable social choice function must maximize a weighted sum of individual utilities plus some alternative-specific constants. Both present assumptions and proof are quite different from Roberts’.\textsuperscript{13}

\textbf{Remark 1} Mezzetti (2004) has shown that efficiency is always (that is, in our context, for any given $\gamma$) attainable with two-stage “generalized Groves” mechanisms where a final allocation is chosen at stage one, and where, subsequently, monetary transfers that depend on the realized ex-post utilities of all agents at that allocation are executed at stage two.\textsuperscript{14} In particular, such mechanisms would require ex-post transfers across all existing partnerships, contingent on the previously realized surplus in each of these pairs. We think that using ex-post information (whether reported or verifiable) to this extent is somewhat unrealistic in the present environment. For example, group manipulations by partners should be an issue for any mechanism that imposes ex-post transfers across pairs. Our fixed-proportion sharing also needs ex-post information, but uses it in a much more limited way to divide surplus within pairs. In particular, there are no contingent payments between pairs or to/from a potential matchmaker after partnerships have formed.

Our second main result deals with the special case where agents’ attributes are one-dimensional. If $n = 1$, then Condition 1 implies that $y \mapsto (\partial_x v)(x, y)$ is either strictly increasing or strictly decreasing. Consequently, $v$ either has strictly increasing differences or strictly decreasing differences in $(x, y)$.\textsuperscript{15} That is, $v$ is either strictly supermodular or strictly submodular. This is the classical one-dimensional assortative/anti-assortative framework à la Becker (1973). We treat here the supermodular case. The submodular one is analogous.

In the one-dimensional supermodular case we find that the class of sharing rules that is compatible with efficient matching is larger, and strictly contains the class of

\textsuperscript{13}Our main technical result is derived by varying a social choice setting with only two alternatives (Roberts studied a single setting with at least three alternatives), surplus may take here general functional forms, and type spaces are arbitrary connected open sets (Roberts has linear utilities and needs an unbounded type space).

\textsuperscript{14}The generalized Groves mechanism has the problem that it does not provide \textit{strict} incentives for truthful reporting of ex-post utilities.

\textsuperscript{15}See also Topkis (1998).
constant rules obtained above.

**Theorem 2** Let $n = 1$, $I, J \geq 2$ and assume that $v$ is strictly supermodular. Then, the efficient matching is implementable in ex-post equilibrium if and only if both $\gamma v$ and $(1 - \gamma)v$ are supermodular.

We derive Theorem 2 by applying a characterization result due to Bergemann and Välimäki (2002). These authors have provided a necessary as well as a set of sufficient conditions for efficient ex-post implementation for one-dimensional types. The logic of our proof is as follows. We first verify that monotonicity in the sense of Definition 4 of Bergemann and Välimäki is satisfied for strictly supermodular match surplus. This is the first part of their set of sufficient conditions (Proposition 3). Then, we show that their necessary condition (Proposition 1) implies that $\gamma v$ and $(1 - \gamma)v$ must be supermodular. Finally, we show that the second part of the sufficient conditions is satisfied as well if $\gamma v$ and $(1 - \gamma)v$ are supermodular.

### 4 Conclusion

We have introduced a novel two-sided matching model with a finite number of agents, two-sided incomplete information, interdependent values, and multi-dimensional attributes. We have shown that fixed-proportion sharing rules are the only ones conducive for efficiency in this setting. While our present result is agnostic about the preferred proportion, augmenting our model with, say, a particular ex-ante investment game will introduce new, additional forces that can be used to differentiate between various constant sharing rules.

### 5 Appendix

We prepare the proof of Theorem 1 by a sequence of lemmas. The key step is Lemma 4 below. It will be very useful to introduce a cross-difference (two-cycle) linear operator $F$, which acts on functions $f : X \times Y \to \mathbb{R}$. The operator $F_f$ has arguments $x^1 \in X^1 = X$, $x^2 \in X^2 = X$, $y^1 \in Y^1 = Y$ and $y^2 \in Y^2 = Y$, and it is defined as follows:

$$F_f(x^1, x^2, y^1, y^2) := f(x^1, y^1) + f(x^2, y^2) - f(x^1, y^2) - f(x^2, y^1).$$

\[\text{We choose superscripts here because } x_1 \text{ is already reserved for the type of employer } e_1, \text{ and so on.}\]
We also define the sets

\[ A := \{(x^1, x^2, y^1, y^2) \in X \times X \times Y \times Y | F_v(x^1, x^2, y^1, y^2) = 0 \}, \]

and

\[ A_0 := \{(x^1, x^2, y^1, y^2) \in A | \nabla F_v(x^1, x^2, y^1, y^2) \neq 0 \}, \]

where

\[ \nabla F_v(x^1, x^2, y^1, y^2) = (\nabla_{x^1} F_v, \nabla_{x^2} F_v, \nabla_{y^1} F_v, \nabla_{y^2} F_v)(x^1, x^2, y^1, y^2). \]

Whenever \( x_1 \neq x_2 \) or \( y_1 \neq y_2 \), Condition 1 implies that \( \nabla F_v(x_1, x_2, y_1, y_2) \neq 0 \). This is repeatedly used below.

**Lemma 1** Let \( n \in \mathbb{N}, I = J = 2 \), and let Condition 1 be satisfied. If the efficient matching is ex-post implementable, then the following implications hold for all \((x_1, x_2, y_1, y_2)\):

\[ F_v(x_1, x_2, y_1, y_2) \geq (\leq) 0 \Rightarrow F_{\gamma v}(x_1, x_2, y_1, y_2) \geq (\leq) 0, \] (1)

\[ F_v(x_1, x_2, y_1, y_2) \geq (\leq) 0 \Rightarrow F_{(1-\gamma)v}(x_1, x_2, y_1, y_2) \geq (\leq) 0. \] (2)

**Proof of Lemma 1.** There are only two alternative matchings, \( m_1 = ((e_1, w_1), (e_2, w_2)) \) and \( m_2 = ((e_1, w_2), (e_2, w_1)) \). Since the efficient matching is ex-post implementable, the taxation principle for ex-post implementation implies that there must be “transfer” functions \( t^{e_1}_{m_1}(x_2, y_1, y_2) \) and \( t^{e_1}_{m_2}(x_2, y_1, y_2) \) for employer \( e_1 \) such that

\[ F_v(x_1, x_2, y_1, y_2) > (\leq) 0 \Rightarrow (\gamma v)(x_1, y_1) + t^{e_1}_{m_1}(x_2, y_1, y_2) \geq (\leq) (\gamma v)(x_1, y_2) + t^{e_1}_{m_2}(x_2, y_1, y_2). \] (3)

For \( y_1 \neq y_2 \), we have \( (\nabla_{x^1} F_v)(x_2, x_2, y_1, y_2) = (\nabla_{x^1} v)(x_2, y_1) - (\nabla_{x^1} v)(x_2, y_2) \neq 0 \) by Condition 1. Hence, in every neighborhood of \( x_1 = x_2 \), there are \( x'_1 \) and \( x''_1 \) such that \( F_v(x'_1, x_2, y_1, y_2) > 0 \) and \( F_v(x''_1, x_2, y_1, y_2) < 0 \). Since \( \gamma v \) is continuous, relation (3) pins down the difference of transfers as:

\[ t^{e_1}_{m_1}(x_2, y_1, y_2) - t^{e_1}_{m_2}(x_2, y_1, y_2) = (\gamma v)(x_2, y_2) - (\gamma v)(x_2, y_1). \]
Plugging this back into (3) yields for all \((x_1, x_2, y_1, y_2)\) with \(y_1 \neq y_2\):

\[
F_v(x_1, x_2, y_1, y_2) > (\leq) 0 \Rightarrow F_{\gamma v}(x_1, x_2, y_1, y_2) > (\leq) 0.
\] (4)

As \(F_v(x_1, x_2, y, y) = F_{\gamma v}(x_1, x_2, y, y) = 0\), relation (4) holds for all \((x_1, x_2, y_1, y_2)\). However, every neighborhood of any \((x_1, x_2, y_1, y_2) \in A\) contains both points at which \(F_v\) is strictly positive and points at which \(F_v\) is strictly negative. Whenever \(x_1 \neq x_2\) or \(y_1 \neq y_2\), this follows immediately from \(\nabla F_v(x_1, x_2, y_1, y_2) \neq 0\). Otherwise, if \(x_1 = x_2\) and \(y_1 = y_2\), one may perturb \(x_2\) by an arbitrarily small amount to some \(x_2'\) (staying in \(A\) since \(y_1 = y_2\)) and apply the argument to \((x_1, x_2', y_1, y_2)\).

Using continuity of \(\gamma v\), (4) may thus be strengthened to (1). A completely analogous argument applies for worker \(w_1\) and yields (2). ■

To prove Theorem 1, we only need local versions of (1) and (2) at profiles where the efficient matching changes. These are available for general \(I, J \geq 2\):

**Lemma 2** Let \(n \in \mathbb{N}\), \(I, J \geq 2\) and let Condition 1 be satisfied. If the efficient matching is ex-post implementable, then for all \((x_1, x_2, y_1, y_2) \in A\), there is an open neighborhood \(U(x_1, x_2, y_1, y_2) \subset X \times X \times Y \times Y\) of \((x_1, x_2, y_1, y_2)\) such that for all \((x_1', x_2', y_1', y_2') \in U(x_1, x_2, y_1, y_2)\):

\[
F_v(x_1', x_2', y_1', y_2') \geq (\leq) 0 \Rightarrow F_{\gamma v}(x_1', x_2', y_1', y_2') \geq (\leq) 0,
\] (5)

\[
F_v(x_1, x_2', y_1', y_2') \geq (\leq) 0 \Rightarrow F_{(1-\gamma)v}(x_1', x_2', y_1', y_2') \geq (\leq) 0.
\] (6)

**Proof of Lemma 2.** Given \((x_1, x_2, y_1, y_2) \in A\), fix the types of all other employers and workers \((x_i\) for \(i \neq 1, 2\), \(y_j\) for \(j \neq 1, 2\) such that there is an open neighborhood \(U(x_1, x_2, y_1, y_2)\) of \((x_1, x_2, y_1, y_2)\) with the following property: for all \((x_1', x_2', y_1', y_2') \in U(x_1, x_2, y_1, y_2)\), the efficient matching for the profile \((x_1', x_2', x_3, \ldots, x_I, y_1', y_2', y_3, \ldots, y_J)\) either matches \(e_1\) to \(w_1\) and \(e_2\) to \(w_2\), or \(e_1\) to \(w_2\) and \(e_2\) to \(w_1\) (depending on the sign of \(F_v(x_1', x_2', y_1', y_2')\)). From here on, the proof parallels the one of Lemma 1. ■

Lemma 2 has the immediate consequence that on \(A_0\), the gradients of \(F_v\), \(F_{\gamma v}\) and \(F_{(1-\gamma)v}\) must all point in the same direction:

**Lemma 3** Let \(n \in \mathbb{N}\), \(I, J \geq 2\) and let Condition 1 be satisfied. If the efficient matching is ex-post implementable, then there is a unique function \(\lambda : A_0 \to [0, 1]\)
Let us spell out the equalities in (7):

\[ \nabla F_{v}(x_{1}, x_{2}, y_{1}, y_{2}) = \lambda(x_{1}, x_{2}, y_{1}, y_{2}) \nabla F_{v}(x_{1}, x_{2}, y_{1}, y_{2}) \]  

(7)

for all \((x_{1}, x_{2}, y_{1}, y_{2}) \in A_{0}\).

**Proof of Lemma 3.** Since \(\nabla F_{v}(x_{1}, x_{2}, y_{1}, y_{2}) \neq 0\) for all \((x_{1}, x_{2}, y_{1}, y_{2}) \in A_{0}\), (5) yields a unique \(\lambda(x_{1}, x_{2}, y_{1}, y_{2}) \geq 0\) with

\[ \nabla F_{v}(x_{1}, x_{2}, y_{1}, y_{2}) = \lambda(x_{1}, x_{2}, y_{1}, y_{2}) \nabla F_{v}(x_{1}, x_{2}, y_{1}, y_{2}). \]

Moreover, \(\nabla F_{v}(0, 1, 2, y_{2}) = (1 - \lambda(x_{1}, x_{2}, y_{1}, y_{2})) \nabla F_{v}(x_{1}, x_{2}, y_{1}, y_{2})\) and (6) therefore implies \(\lambda(x_{1}, x_{2}, y_{1}, y_{2}) \in [0, 1]\). □

The crucial step in the proof follows now. It shows that for \(n \geq 2\) the function \(\lambda\) must be constant. This constant corresponds then to a particular fixed-proportion sharing rule.

**Lemma 4** Let \(n \geq 2, I, J \geq 2\) and let Condition 1 be satisfied. Then the function \(\lambda\) from Lemma 3 must be constant: there is a \(\lambda_{0} \in [0, 1]\) such that \(\lambda \equiv \lambda_{0}\).

**Proof of Lemma 4.** Let us spell out the equalities in (7):

\[
(\nabla X v)(x_{1}, y_{1}) - (\nabla X v)(x_{1}, y_{2}) = \lambda(x_{1}, x_{2}, y_{1}, y_{2})((\nabla X v)(x_{1}, y_{1}) - (\nabla X v)(x_{1}, y_{2}))
\]

\[
(\nabla X v)(x_{2}, y_{2}) - (\nabla X v)(x_{2}, y_{1}) = \lambda(x_{1}, x_{2}, y_{1}, y_{2})((\nabla X v)(x_{2}, y_{2}) - (\nabla X v)(x_{2}, y_{1}))
\]

\[
(\nabla Y v)(x_{1}, y_{1}) - (\nabla Y v)(x_{2}, y_{1}) = \lambda(x_{1}, x_{2}, y_{1}, y_{2})((\nabla Y v)(x_{1}, y_{1}) - (\nabla Y v)(x_{2}, y_{1}))
\]

\[
(\nabla Y v)(x_{2}, y_{2}) - (\nabla Y v)(x_{2}, y_{1}) = \lambda(x_{1}, x_{2}, y_{1}, y_{2})((\nabla Y v)(x_{2}, y_{2}) - (\nabla Y v)(x_{2}, y_{1})).
\]

(8)

Given any \((x_{1}, x_{2}, y_{1}, y_{2}) \in A_{0}\), one obtains the same system of equations at \((x_{2}, x_{1}, y_{1}, y_{2}) \in A_{0}\), albeit for \(\lambda(x_{2}, x_{1}, y_{1}, y_{2})\). Thus, the function \(\lambda\) is symmetric with respect to \(x_{1}\) and \(x_{2}\). Similarly, it is symmetric with respect to \(y_{1}\) and \(y_{2}\). Next, for given \(x_{1} \in X\) and \(y_{1} \neq y_{2}\), the vectors in the first equation of (8) (with \((\nabla X v)(x_{1}, y_{1}) - (\nabla X v)(x_{1}, y_{2}) \neq 0\) on the right hand side) do not depend on how \((x_{1}, y_{1}, y_{2})\) is completed by \(x_{2}\) to yield a full profile that lies in \(A_{0}\). Consequently, \(\lambda(x_{1}, x_{2}, y_{1}, y_{2}) = \lambda(x_{1}, x_{1}, y_{1}, y_{2})\) for all these possible choices.

We next show that for a given \(x_{1}\), \(\lambda\) does in fact not depend on \(y_{1}\) and \(y_{2}\) as long as \(y_{1} \neq y_{2}\). To this end, start with any \(x_{1} \in X\) and \(y_{1} \neq y_{2}\). We will show that for
all \( y'_2 \neq y_1 \) it holds
\[
\lambda(x_1, x_1, y_1, y_2) = \lambda(x_1, x_1, y_1, y'_2). \tag{9}
\]

Then, by symmetry of \( \lambda \), \( \lambda(x_1, x_1, y_1, y_2) = \lambda(x_1, x_1, y'_2, y_1) \), and repeating the argument will yield that \( \lambda \) is indeed independent of \( y_1 \) and \( y_2 \) as long as \( y_1 \neq y_2 \).

So, let us prove (9). Using the first equation of (8), we have:

\[
\begin{align*}
\lambda(x_1, x_1, y_1, y_2) & ((\nabla X v)(x_1, y_1) - (\nabla X v)(x_1, y_2)) \\
& = ((\nabla X \gamma v)(x_1, y_1) - (\nabla X \gamma v)(x_1, y'_2)) + ((\nabla X \gamma v)(x_1, y'_2) - (\nabla X \gamma v)(x_1, y_2)) \\
& = \lambda(x_1, x_1, y_1, y'_2)((\nabla X v)(x_1, y_1) - (\nabla X v)(x_1, y'_2)) \\
& + \lambda(x_1, x_1, y'_2, y_2)((\nabla X v)(x_1, y'_2) - (\nabla X v)(x_1, y_2)).
\end{align*}
\]

It follows that
\[
\begin{align*}
(\lambda(x_1, x_1, y_1, y'_2) - \lambda(x_1, x_1, y_1, y_2)) & ((\nabla X v)(x_1, y_1) - (\nabla X v)(x_1, y'_2)) \\
& + (\lambda(x_1, x_1, y'_2, y_2) - \lambda(x_1, x_1, y_1, y_2))((\nabla X v)(x_1, y'_2) - (\nabla X v)(x_1, y_2)) \\
& = 0. \tag{10}
\end{align*}
\]

Two cases must now be distinguished.

**Case 1:** \((\nabla X v)(x_1, y_1) - (\nabla X v)(x_1, y'_2)\) and \((\nabla X v)(x_1, y'_2) - (\nabla X v)(x_1, y_2)\) are linearly independent. Then, it follows from (10) that \( \lambda(x_1, x_1, y_1, y'_2) = \lambda(x_1, x_1, y_1, y_2) \).

**Case 2:** \((\nabla X v)(x_1, y_1) - (\nabla X v)(x_1, y'_2)\) and \((\nabla X v)(x_1, y'_2) - (\nabla X v)(x_1, y_2)\) are linearly dependent. In this case, pick some \( y''_2 \in Y \) such that \((\nabla X v)(x_1, y_1) - (\nabla X v)(x_1, y'_2)\) and \((\nabla X v)(x_1, y'_2) - (\nabla X v)(x_1, y_2)\) are linearly independent. This is always possible since \((\nabla X v)(x_1, \cdot)\) maps open neighborhoods of \( y_1 \) one-to-one into \( \mathbb{R}^n \), and since for \( n \geq 2 \), there is no one-to-one continuous mapping from an open set in \( \mathbb{R}^n \) to the real line \( \mathbb{R} \).\(^{17}\)

From Case 1, we obtain \( \lambda(x_1, x_1, y_1, y''_2) = \lambda(x_1, x_1, y_1, y_2) \). Since \((\nabla X v)(x_1, y_1) - (\nabla X v)(x_1, y'_2)\) and \((\nabla X v)(x_1, y'_2) - (\nabla X v)(x_1, y''_2)\) are also linearly independent, we then get \( \lambda(x_1, x_1, y_1, y'_2) = \lambda(x_1, x_1, y_1, y''_2) \), and hence (9) follows.

The third equation of (8) may be now used in an analogous way to show that for a given \( y_1 \), \( \lambda(x_1, x_2, y_1, y_1) \) does not depend on \( x_1 \) and \( x_2 \), as long as \( x_1 \neq x_2 \).

The final ingredient is the following observation: for every \((x_1, x_1, y_1, y_2) \in A_0,\)

\(^{17}\)This is a special case of Brouwer’s (1911) classical dimension preservation result: For \( k < m \), there is no one-to-one, continuous function from a non-empty open set \( U \) of \( \mathbb{R}^m \) into \( \mathbb{R}^k \).
there is a \( x_2 \neq x_1 \) with \( (x_1, x_2, y_1, y_2) \in A_0 \). Indeed, \( (\nabla X^2 F_e)(x_1, x_1, y_1, y_2) \neq 0 \), so that the set of \( x_2 \) for which \( (x_1, x_2, y_1, y_2) \in A_0 \) is given locally (in a neighborhood of \( x_2 = x_1 \)) by a differentiable manifold of dimension \( n - 1 \). Since \( n \geq 2 \), this manifold must contain points other than \( x_1 \). A similar argument applies to \( (x_1, x_2, y_1, y_1) \in A_0 \).

To conclude the proof, we show that \( \lambda \) is constant on \( \{(x_1, x_2, y_1, y_2) \in A_0 | x_1 \neq x_2 \text{ and } y_1 \neq y_2 \} \). This set is non-empty by the previous observation (and we have already seen that \( \lambda(x_1, x_2, y_1, y_2) = \lambda(x_1, x_1, y_1, y_2) \) and \( \lambda(x_1, x_2, y_1, y_2) = \lambda(x_1, x_2, y_1, y_1) \), so that \( \lambda \) is constant on all of \( A_0 \) then). Given any \( (x_1, x_2, y_1, y_2), (x'_1, x'_2, y'_1, y'_2) \in A_0 \) with \( x_1 \neq x_2, y_1 \neq y_2, x'_1 \neq x'_2 \) and \( y'_1 \neq y'_2 \), we have:

\[
\lambda(x_1, x_2, y_1, y_2) = \lambda(x_1, x_1, y_1, y_2) = \lambda(x_1, x'_1, y'_1, y'_2)
\]

\[
= \lambda(x_1, x'_2, y'_1, y'_2) = \lambda(x_1, x'_2, y'_1, y'_1)
\]

\[
= \lambda(x'_1, x'_2, y'_1, y'_1) = \lambda(x'_1, x'_2, y'_1, y'_2),
\]

where \( x'_2 \neq x_1 \) is any feasible profile completion for \( (x_1, y'_1, y'_2) \).

We are now finally ready to prove Theorem 1.

**Proof of Theorem 1. ii) \( \Rightarrow \) i)**: Consider the case \( I \leq J \). As in the proof of Lemma 1, we make use of the “taxation principle” for ex-post implementation. For employer \( e_i \), and matching \( m \in M \) define \( t^e_i(x_{-i}, y_1, \ldots, y_I) := \lambda_0 \sum_{l \neq i} v(x_l, y_{m(l)}) - h(y_{m(i)}) \).

Then, \((\gamma v)(x_i, y_{m(i)}) + t^e_i(x_{-i}, y_1, \ldots, y_I) = \lambda_0 \sum_{l=1}^I v(x_l, y_{m(l)}) + g(x_i)\), so that it is optimal for \( e_i \) to select a matching that maximizes aggregate welfare. Note that strict incentives for truth-telling can be provided only if \( \lambda_0 > 0 \). For worker \( w_j \), define

\[
t^{w_j}_m(x_1, \ldots, x_I, y_{-j}) := (1 - \lambda_0) \sum_{k \in m(I), k \neq j} v(x_{m^{-1}(k)}, y_k)
\]

\[
+ g(x_{m^{-1}(j)}) \mathbf{1}_{j \in m(I)} - h(y_j) \mathbf{1}_{j \notin m(I)}.
\]

Here, \( \mathbf{1}_{j \in m(I)} = 1 \) if \( j \in m(I) \), and \( \mathbf{1}_{j \notin m(I)} = 0 \) otherwise. Note that if \( I = J \), then \( j \in m(I) \) for all possible matchings \( m \), so that the final (\( y_j \)-dependent) term always vanishes. If \( I < J \), then \( h \) is constant by assumption, and the transfer does not depend on \( y_j \). It follows that if \( w_j \) is matched in \( m \), his utility is \(((1 - \gamma) v)(x_{m^{-1}(j)}, y_j) + t^{w_j}_m(x_1, \ldots, x_I, y_{-j}) = (1 - \lambda_0) \sum_{k \in m(I)} v(x_{m^{-1}(k)}, y_k) - h(y_j)\). Otherwise, his utility is just \( t^{w_j}_m(x_1, \ldots, x_I, y_{-j}) = (1 - \lambda_0) \sum_{k \in m(I)} v(x_{m^{-1}(k)}, y_k) - h(y_j)\). Hence, it is optimal for \( w_j \) to select a matching that maximizes aggregate welfare, and strict incentives.

17
for truth-telling can be provided only if \( \lambda_0 < 1 \). This proves i) for \( I \leq J \). The proof for the case \( I \geq J \) is completely analogous.

i) \( \Rightarrow \) ii): By Lemma 4, there is a \( \lambda_0 \in [0, 1] \) such that for all \( x \in X, y_1, y_2 \in Y \) with \( y_1 \neq y_2 \) it holds (the profile may be completed to lie in \( A_0 \), e.g. by \( x' = x \)):
\[
(\nabla_X \gamma v)(x, y_1) - (\nabla_X \gamma v)(x, y_2) = \lambda_0 ((\nabla_X v)(x, y_1) - (\nabla_X v)(x, y_2)).
\]

Integrating along any path from \( x_2 \) to \( x_1 \) (\( X \) is open and connected in \( \mathbb{R}^n \), hence path-connected) yields \( F_{\gamma v}(x_1, x_2, y_1, y_2) = \lambda_0 F_v(x_1, x_2, y_1, y_2) \). Hence, by linearity of the operator \( F \), we obtain that \( F(\gamma - \lambda_0)v \equiv 0 \). A function of two variables has vanishing cross differences if and only if it is additively separable, so that we can write \( (\gamma v)(x, y) = \lambda_0 v(x, y) + g(x) + h(y) \). This concludes the proof for the case where \( I = J \).

It remains to prove that \( h \) must be constant if \( I < J \) (the proof that \( g \) must be constant when \( I > J \) is analogous). Given \( y_1 \in Y \), Condition 1 implies that \( (\nabla_Y v)(\cdot, y_1) \) vanishes at most in one point. Pick then any \( x_1 \in X \) with \( (\nabla_Y v)(x_1, y_1) \neq 0 \). Set \( y_2 = y_1 \) and complete the type profile for \( (i \neq 1, j \neq 1, 2) \) such that, for an open neighborhood \( U \) of \( (y_1, y_1) \), the efficient matching changes only with respect to the partner of \( e_1 \); either \( w_1 \) is matched to \( e_1 \) and \( w_2 \) remains unmatched, or \( w_2 \) is matched to \( e_1 \) and \( w_1 \) remains unmatched. For \( (y_1', y_2') \in U \), it follows that \( v(x_1, y_1') - v(x_1, y_2') \geq (\leq) 0 \) implies \( ((1 - \gamma)v)(x_1, y_1') - ((1 - \gamma)v)(x_1, y_2') \geq (\leq) 0 \). Hence, there is a \( \mu(x_1, y_1) \geq 0 \) such that
\[
(1 - \lambda_0)(\nabla_Y v)(x_1, y_1) - (\nabla_Y h)(y_1) = \mu(x_1, y_1)(\nabla_Y v)(x_1, y_1).
\]

In other words, \( (\nabla_Y h)(y_1) \) and \( (\nabla_Y v)(x_1, y_1) \) are linearly dependent. Finally, let \( x_1 \) vary and note that, by Condition 1, the image of \( (\nabla_Y v)(\cdot, y_1) \) cannot be concentrated on a line (recall footnote 17). Thus, we obtain that \( (\nabla_Y h)(y_1) = 0 \). Since \( y_1 \) was arbitrary and \( Y \) is connected, it follows that the function \( h \) must constant. ■

**Proof of Theorem 2.** Let \( I \leq J \) (the proof for \( I \geq J \) is analogous). Consider some \( i \in \mathcal{I} \) and a given, fixed type profile for all other agents \((x_{-i}, y_1, \ldots, y_J)\). Given any such type profile, we re-order the workers and employers other than \( i \) such that \( x^{(1)} \geq \ldots \geq x^{(I-1)} \) and \( y^{(1)} \geq \ldots \geq y^{(J)} \).

We now verify the monotonicity condition identified by Bergemann and Välimäki.\(^{18}\)

\(^{18}\)We only verify it for type profiles for which all these inequalities are strict. When some types co-
This requires that the set of types of agent $i$ for which a particular social alternative is efficient forms an interval. Let then $m_i$, $k = 1, \ldots, I$ denote the matching that matches $x^{(l)}$ to $y^{(l)}$ for $l = 1, \ldots, k - 1$, $x_i$ to $y^{(k)}$ and $x^{(l)}$ to $y^{(l+1)}$ for $l = k, \ldots, I - 1$. Then, for $k = 2, \ldots, I - 1$ it holds that the set

$$\{x_i \in X \mid u_{m_k}(x_1, \ldots, x_I, y_1, \ldots, y_J) \geq u_m(x_1, \ldots, x_I, y_1, \ldots, y_J), \forall m \in M\}$$

is simply $[x^{(k)}, x^{(k-1)}]$. For $k = I$ the set is $(\inf X, x^{(I-1)})$, and for $k = 1$ it is $[x^{(1)}, \sup X)$. Monotonicity for workers $j$ is verified in the same way.

Next, the necessary condition of Bergemann and Välimäki, spelled out for our matching model, requires that at all “switching points” $x_i = x^{(k-1)}$ where the efficient allocation changes, it also holds that

$$\frac{\partial}{\partial x_i}((\gamma v)(x_i, y^{(k-1)}) - (\gamma v)(x_i, y^{(k)})) \geq 0.$$ 

Given $x_i$ and $y' > y$ we can always complete these to a full type profile such that $x_i$ is a change point at which the efficient match for $x_i$ switches from $y$ to $y'$. Hence $\frac{\partial}{\partial x}((\gamma v)(x, y') - (\gamma v)(x, y)) \geq 0$ for all $x$ and $y' > y$. So, $\gamma v$ must have increasing differences, i.e. it is supermodular. Since $\frac{\partial}{\partial x_i}((\gamma v)(x_i, y^{(k-1)}) - (\gamma v)(x_i, y^{(k)})) \geq 0$ is satisfied for all $x_i \in X$ (not just at switching points!), the second part of the sufficient conditions of Bergemann and Välimäki is satisfied. The argument for workers (yielding supermodularity of $(1-\gamma)v$) is analogous. This completes the proof. ■

References


incide, it is still straightforward to verify monotonicity but we do not spell out the more cumbersome case distinctions here.


