A Fundamental Question in Mechanism Design

Claim (Manelli &Vincent, *Econometrica* 2010): "We prove that in the independent private-values model with linear utility, the outcome - in terms of interim expected probabilities of trade and interim expected transfers - of any Bayesian incentive compatible mechanism can also be obtained via a dominant strategy mechanism...The equivalence result holds, in particular, in many commonly studied auction models"

A Problem in Discrete Tomography

Problem

When does a $0-1$ matrix with given row and column sums exist ?

- Consider row sum $(3, 2, 2, 1, 1)$ and two different column sums:

1	1	0	0	1	**3**
1	1	0	0	0	**2**
1	1	0	0	0	**2**
1	0	0	0	0	**1**
0	1	0	0	0	**1**
4	**4**	**0**	**0**	**1**	

1	2!	0	0	0	**3**
1	1	0	0	0	**2**
1	1	0	0	0	**2**
1	0	0	0	0	**1**
1	0	0	0	0	**1**
5	**4**	**0**	**0**	**0**	

- Matrix exist if the vector of column sums is "less diverse" than the vector $(5, 3, 1, 0, 0)$.
- See Gale (1957), and Ryser (1957) for the general result. Variations (continuous case, densities) are in Kellerer (1961) and Strassen (1965).

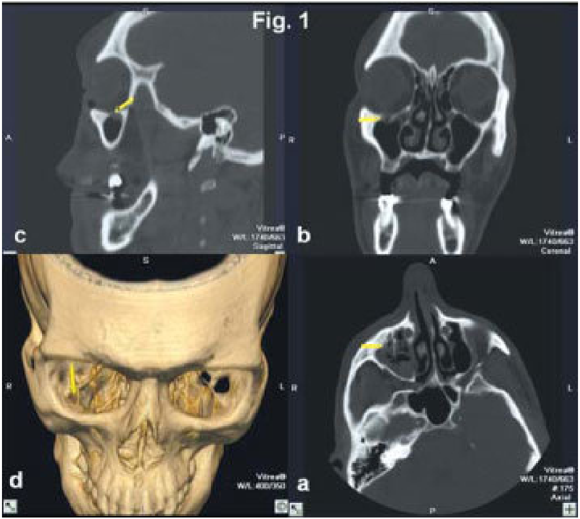

FIGURE 1- In this sequence of images (axial (a), coronal (b), sagittal (c) and 3D-CT (d)), it may be observed an anterior inferior orbit wall fracture (arrows). An arrow is simultaneously pointed by the software in all the images. This fracture cannot be detect in axial image

The Monotone Lift

Problem

When unique reconstruction is not possible, are there solutions with special properties ?

Theorem (Gutmann et al. (1991))

Let $\phi = \phi(x_1, x_2, ..x_n)$ *be measurable on* $[0,1]^n$ *with* $0 \leq \phi \leq 1$*.Assume that the one-dimensional* **marginals**

$$\Phi_i(x_i) = \int \phi(x_1, x_2, ..x_n) dx_{-i}$$

are **non-decreasing** *in* x_i, $i = 1, 2, ..n$. *Then there exists* ψ *measurable on* $[0,1]^n$ *such that* $0 \leq \psi \leq 1$, ψ *has the* **same marginals** *as* ϕ, *and moreover,* ψ *is* **non-decreasing in each coordinate**.

Monotone Lift: Example

Example

$$\phi = \begin{array}{cccc} 2 & 4 & 4 & \mathbf{10} \\ 4 & 2 & 6 & \mathbf{12} \\ 4 & 6 & 4 & \mathbf{14} \\ \mathbf{10} & \mathbf{12} & \mathbf{14} & \end{array} \qquad \Longrightarrow \qquad \psi = \begin{array}{cccc} 2 & 4 & 4 & \mathbf{10} \\ 4 & 4 & 4 & \mathbf{12} \\ 4 & 4 & 6 & \mathbf{14} \\ \mathbf{10} & \mathbf{12} & \mathbf{14} & \end{array}$$

- Note that $\sum_{i,j}(\psi_{ij})^2 \leq \sum_{i,j}(\phi_{ij})^2$.

The Independent Private Values Model with Linear Utility

- K social alternatives and N agents. The utility of agent i in alternative k is given by $a_i^k x_i + c_i^k + t_i$ where $x_i \in [0, 1]$ is agent i's private type, where $a_i^k, c_i^k \in \mathbb{R}$ with $a_i^k \geq 0$, and where $t_i \in \mathbb{R}$ is a monetary transfer.
- Types are drawn independently of each other, according to strictly increasing distributions F_i. Type x_i is private information of agent i.
- Manelli and Vincent assume: $K = N$; $a_i^i = 1$, $a_i^j = 0$ for any $j \neq i$; $c_i^k = 0$ for any i, k.

Incentive Compatible Mechanisms I

Definition

A direct revelation mechanism (DRM) **M** is given by K functions $q^k : [0,1]^N \longrightarrow [0,1]$ and N functions $t_i : [0,1]^N \longrightarrow \mathbb{R}$ where $q^k(x_1, ..., x_N)$ is the probability with which alternative k is chosen, and $t_i(x_1, ..., x_N)$ is the transfer to agent i if the agents report types $x_1, ..., x_N$.

Definition

A DRM **M** is *Dominant-Strategy Incentive Compatible* (DIC) if truth-telling constitutes a dominant strategy equilibrium in the game defined by **M** and the given utility functions. A DRM **M** is *Bayes-Nash Incentive Compatible* (BIC) if truth-telling constitutes a Bayes-Nash equilibrium in the game defined by **M** and the given utility functions.

Incentive Compatible Mechanisms II

Fact

A necessary condition for **M** *to be DIC is that, for each agent* i **,** *and for any signals of others, the function* $\sum_{k=1}^{K} a_i^k q^k(x_1, ..., x_N)$ *is non-decreasing in* x_i*. Moreover, any* K *functions* q^k *that satisfy this condition are part of a DIC mechanism.*

Fact

A necessary condition for **M** *to be BIC is that, for each agent* i*, the function* $\sum_{k=1}^{K} a_i^k Q_i^k(x_i)$ *is non-decreasing, where*

$$\forall i, k, \; Q_i^k(\widehat{x}_i) = \int_{[0,1]^{N-1}} q^k(x_1, ..., x_i, \widehat{x}_i, x_{i+1}, ..., x_N) dF_{-i},$$

is the expected probability that alternative k *is chosen if agents* $j \neq i$ *report truthfully while agent* i *reports type* $\widehat{x}_i$*.Moreover any* K *functions* q^k *that satisfy this condition are part of a BIC mechanism***.**

Equivalent Mechanisms

Definition

1. Two mechanisms $\mathbf{M}$ and $\widetilde{\mathbf{M}}$ are *P-equivalent* if, for each i, k and x_i, it holds that $Q_i^k(x_i) = \widetilde{Q}_i^k(x_i)$, where Q_i^k and $\widetilde{Q}_i^k$ are the conditional expected probabilities associated with $\mathbf{M}$ and $\widetilde{\mathbf{M}}$, respectively.
2. Two mechanisms $\mathbf{M}$ and $\widetilde{\mathbf{M}}$ are U-equivalent if they provide the same interim utilities for each agent i and each type x_i of agent i.

- For each agent i, interim utility is obtained (up to a constant) by integrating the function $\sum_{k=1}^{K} a_i^k Q_i^k(x_i)$ with respect to x_i - this is the **Payoff Equivalence Theorem**. Thus P-equivalence implies U-equivalence.

P- and U-Equivalence for 2 Alternatives

- Since $q^2(x_1, ..., x_N) = 1 - q^1(x_1, ..., x_N)$, we have

$$\sum_{k=1}^{2} a_i^k Q_i^k(x_i) = a_i^2 + (a_i^1 - a_i^2)Q_i^1(x_i),$$

and therefore U-equivalence implies P-equivalence (the two notions coincide).

Theorem

Assume that $K = 2$. Then for any BIC mechanism there exists a P-equivalent (and thus U-equivalent) DIC mechanism.

U-Equivalence for Symmetric Settings

Theorem

Assume that $a_i^k = a_j^k = a^k$ for all k, i, j, and that $F_i = F$ for all i. Moreover, assume that $0 = a^1 \leq a^2 \leq \cdots \leq a^K = 1$. Then for any symmetric, BIC mechanism there exists an U-equivalent symmetric DIC mechanism

- Proof shows how to achieve U-equivalence using only the 2 alternatives with highest and lowest slope, respectively. Thus, U-equivalence does not necessarily ensure that the ex-ante probabilities of different alternatives are preserved.