Part A

Exercises:

1. Solve Exercise 21.D.5 in MWG.

2. Solve Exercise 21.D.10 in MWG.

3. **Gibbard-Satterthwaite** (harder)

   Consider the Gibbard-Satterthwaite setting from the lecture: \( X \) denotes the set of alternatives, \(|X| \geq 3\), and \( R_i = \mathcal{P} \) for all agents \( i \), where \( \mathcal{P} \) denotes the set of all rational preference profiles without indifferences.

   Given a preference \( P_i \) of agent \( i \), denote by \( r_1(P_i) \) the alternative that is ranked first under preference \( P_i \) (and analogously for \( r_k(P_i) \)).

   (a) Show: Suppose the social choice function \( f \) is dominant-strategy incentive compatible. Then \( f \) is monotonic. (For the definition of monotonicity, see Definition 21.E.4 on page 808 in MWG.)

   (b) Show: Suppose the social choice function \( f \) is dominant-strategy incentive compatible and onto. Then \( f \) is Pareto efficient.

   Suppose from now on that there are only two agents, 1 and 2, and consider a social choice function \( f \) that is dominant-strategy incentive compatible and onto.

   (c) Let \( x, y \in X \), \( x \neq y \) and consider a preference profile \( (\bar{P}_1, \bar{P}_2) \) such that \( r_1(\bar{P}_1) = r_2(\bar{P}_2) = x \), \( r_2(\bar{P}_1) = r_1(\bar{P}_2) = y \), and suppose that \( f(\bar{P}_1, \bar{P}_2) = x \). Show that \( f(P_1, P_2) = x \) for all \((P_1, P_2)\) such that \( r_1(P_1) = x \).

   (d) Repeating the previous argument for all pairs of alternatives yields two sets \( X_1 \subseteq X \) and \( X_2 \subseteq X \) such that \( z \in X_1 \) if and only if \( r_1(P_1) = z \) implies \( f(P_1, P_2) = z \) (and \( z \in X_2 \) if and only if \( r_1(P_2) = z \) implies \( f(P_1, P_2) = z \)).

   Show that \( X_1 = X \). What do you conclude?

   (Hint: Use the previous part! Argue that \( X_0 = X \setminus \{X_1 \cup X_2\} \) contains at most one alternative and hence \( X_1 \cup X_2 \) contains at least two distinct alternatives. Argue then that either \( X_1 \) or \( X_2 \) must be empty. Using the previous part you can show now that \( X_0 \) must also be empty.)

Part B

4. Suppose there are 7 voters and two alternatives, \( A \) and \( B \). Voter \( i \) values alternative \( A \) at 0 and alternative \( B \) at \( \theta_i \), where \( \theta_i \) is uniformly and independently distributed on \([-2, 1]\) for each \( i \).

Which majority requirement maximizes utilitarian welfare?

5. Consider the following marriage market with four men and four women.

Preferences are strict and given by

\[
\begin{align*}
    m_1 : & \quad w_3, \quad w_2, \quad w_1, \quad w_4 \\
    m_2 : & \quad w_2, \quad w_1, \quad w_3, \quad w_4 \\
    m_3 : & \quad w_3, \quad w_4, \quad w_1, \quad w_2 \\
    m_4 : & \quad w_4, \quad w_3, \quad w_2, \quad w_1 \\
    w_1 : & \quad m_4, \quad m_3, \quad m_2, \quad m_1 \\
    w_2 : & \quad m_3, \quad m_4, \quad m_1, \quad m_2 \\
    w_3 : & \quad m_2, \quad m_1, \quad m_4, \quad m_3 \\
    w_4 : & \quad m_2, \quad m_3.
\end{align*}
\]

In the following, you may use without proof that

\[
\mu = \begin{pmatrix}
    w_1 & w_2 & w_3 & w_4 \\
m_3 & m_1 & m_4 & m_2
\end{pmatrix}
\]
is a stable matching. You may use all results from the lecture without proof.

(a) Show that

\[ \mu_2 = \begin{array}{cccc}
    w_1 & w_2 & w_3 & w_4 \\
    m_2 & m_4 & m_1 & m_3
\end{array} \]

is a stable matching.

(b) Find the men-optimal stable matching \( \mu_M \) and the women-optimal stable matching \( \mu_W \).