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Part A

Exercises:

1. Solve Exercise 21.D.5 in MWG.

2. Solve Exercise 21.D.10 in MWG.

3. *Gibbard-Satterthwaite* (harder)

   Consider the Gibbard-Satterthwaite setting from the lecture: X denotes the set of alternatives, $|X| \geq 3$, and $\mathcal{R}_i = \mathcal{P}$ for all agents i, where $\mathcal{P}$ denotes the set of all rational preference profiles without indifferences.

   Given a preference P_i of agent i, denote by $r_1(P_i)$ the alternative that is ranked first under preference P_i (and analogously for $r_k(P_i)$).

   (a) Show: Suppose the social choice function f is dominant-strategy incentive compatible. Then f is monotonic. (For the definition of monotonicity, see Definition 21.E.4 on page 808 in MWG.)

   (b) Show: Suppose the social choice function f is dominant-strategy incentive compatible and onto. Then f is Pareto efficient.

   Suppose from now on that there are only two agents, 1 and 2, and consider a social choice function f that is dominant-strategy incentive compatible and onto.

   (c) Let $x, y \in X$, $x \neq y$ and consider a preference profile $(\overline{P}_1, \overline{P}_2,)$ such that $r_1(\overline{P}_1) = r_2(\overline{P}_2) = x$, $r_2(\overline{P}_1) = r_1(\overline{P}_2) = y$, and suppose that $f(\overline{P}_1, \overline{P}_2,) = x$. Show that $f(P_1, P_2) = x$ for all (P_1, P_2) such that $r_1(P_1) = x$.

   (d) Repeating the previous argument for all pairs of alternatives yields two sets $X_1 \subseteq X$ and $X_2 \subseteq X$ such that $z \in X_1$ if and only if $r_1(P_1) = z$ implies $f(P_1, P_2) = z$ (and $z \in X_2$ if and only if $r_1(P_2) = z$ implies $f(P_1, P_2) = z$).

   Show that $X_1 = X$. What do you conclude?

   (Hint: Use the previous part! Argue that $X_0 = X \setminus \{X_1 \cup X_2\}$ contains at most one alternative and hence $X_1 \cup X_2$ contains at least two distinct alternatives. Argue then that either X_1 or X_2 must be empty. Using the previous part you can show now that X_0 must also be empty.)

Part B

4. Suppose there are 7 voters and two alternatives, A and B. Voter i values alternative A at 0 and alternative B at θ_i, where θ_i is uniformly and independently distributed on $[-2, 1]$ for each i.

   Which majority requirement maximizes utilitarian welfare?

5. Consider the following marriage market with four men and four women.
   Preferences are strict and given by

m_1 :	w_1,	w_2,	w_3,	w_4	w_1 :	m_4,	m_3,	m_2,	m_1
m_2 :	w_2,	w_1,	w_4,	w_3	w_2 :	m_3,	m_4,	m_1,	m_2
m_3 :	w_3,	w_4,	w_1,	w_2	w_3 :	m_2,	m_1,	m_4,	m_3
m_4 :	w_4,	w_3,	w_2,	w_1	w_4 :	m_2,	m_3.		

   In the following, you may use without proof that

$$\mu_1 = \begin{matrix} w_1 & w_2 & w_3 & w_4 \\ m_3 & m_1 & m_4 & m_2 \end{matrix}$$

is a stable matching. You may use all results from the lecture without proof.

(a) Show that

$$\mu_2 = \begin{matrix} w_1 & w_2 & w_3 & w_4 \\ m_2 & m_4 & m_1 & m_3 \end{matrix}$$

is a stable matching.

(b) Find the men-optimal stable matching μ_M and the women-optimal stable matching μ_W.