**Microeconomics II (PhD)**
**Tutorial 4, June 6** akleiner@uni-bonn.de

# Part A

You can use the following envelope theorem in the next exercise.

**Theorem** (Milgrom and Segal, 2002).
*Let $X$ be an arbitrary set, $T = [\underline{t}, \overline{t}]$,[1] and $f : X \times T \to \mathbb{R}$. Denote*

$$V(t) = \sup_{x \in X} f(x,t) \tag{1}$$

$$X^*(t) = \{x \in X | f(x,t) = V(t)\}. \tag{2}$$

*Suppose that $f(x,\cdot)$ is differentiable for all $x \in X$, $f_t(x,\cdot)$ is uniformly bounded and that $X^*(t) \neq \emptyset$ for almost all $t$. Then for any selection $x^*(t) \in X^*(t)$,*

$$V(t) = V(\underline{t}) + \int_{\underline{t}}^{t} f_t(x^*(s), s) ds. \tag{3}$$

1. Consider the general mechanism design setting from the lecture, where $v_i(k, \theta_i)$ denotes the value of allocation $k$ to agent $i$ with type $\theta_i$. Suppose that $\Theta_i = [\underline{\theta}_i, \overline{\theta}_i] \subset \mathbb{R}$ and that $v_i$ is differentiable in $\theta_i$ for all $k$ and the derivative is uniformly bounded. Given a direct revelation mechanism $(k, t)$, let $U_i(\theta) = v_i(k(\theta), \theta_i) + t_i(\theta)$ be the utility of agent $i$ if $\theta$ is the profile of types and all agents report truthfully.

   (a) Show that if the direct revelation mechanism $(k, t)$ is implementable in dominant strategies, then

   $$U_i(\theta) = U_i(\underline{\theta}_i, \theta_{-i}) + \int_{\underline{\theta}_i}^{\theta_i} \frac{\partial v_i(k(s, \theta_{-i}), s)}{\partial \theta_i} ds. \tag{ICFOC}$$

   Suppose that $v_i(k, \theta_i)$ has the single-crossing property: $\frac{\partial^2 v_i(k,\theta_i)}{\partial k \, \partial \theta_i}$ exists and is stricly positive for all $k \in K$ and $\theta_i \in \Theta_i$.

   (b) Show that if the direct revelation mechanism $(k, t)$ is implementable in dominant strategies, then $k(\theta_i, \theta_{-i})$ is weakly increasing in $\theta_i$ for all $\theta_{-i}$.

   (c) Show that any monotone mechanism that satisfies (ICFOC) is implementable in dominant strategies.

   (d) Discuss the relation of these results to the result that you saw in the lecture.

   (e) Show: If a direct revelation mechanism implements the value-maximizing allocation rule in dominant strategies, then it is a VCG mechanism.

# Part B

2. Suppose there is one agent, three potential types $(\theta^1, \theta^2, \theta^3)$ and three alternatives $(a, b, c)$. The valuation the agent has for an alternative given his type is given by the following matrix:

   | | $\theta^1$ | $\theta^2$ | $\theta^3$ |
   |---|---|---|---|
   | a | 0 | -1 | x |
   | b | 1 | 0 | -1 |
   | c | -1 | 1 | 0 |

   Consider the function $k'$ such that $k'(\theta^1) = a$, $k'(\theta^2) = b$, and $k'(\theta^3) = c$.

   (a)

[1] This result holds more generally, for example if $T \subset \mathbb{R}^n$ is convex.

**Definition 1.** *A decision rule $k$ is* weakly monotone *if for all $\theta^i, \theta^j$,*

$$v(k(\theta^i), \theta^i) - v(k(\theta^j), \theta^i) \geq v(k(\theta^i), \theta^j) - v(k(\theta^j), \theta^j).$$

Suppose $x = 1$. Is $k'$ weakly monotone? Is it implementable in the sense that there is a payment rule $t$ such that $(k', t)$ is incentive compatible? How does this relate to the result you saw in the lecture?

(b)

**Definition 2.** *A decision rule $k$ is* cyclically monotone *if for every sequence of types of length $l \in \mathbb{N}$, $(\theta^1, \theta^2, ..., \theta^l)$, with $\theta^l = \theta^1$, we have*

$$\sum_{\kappa=1}^{l-1} v(k(\theta^\kappa), \theta^{\kappa+1}) - v(k(\theta^\kappa), \theta^\kappa) \leq 0.$$

Show that every implementable decision rule $k$ is cyclically monotone.

(c) For which values of $x$ is $k'$ cyclically monotone?

3. There is one seller with two objects, and one buyer. The seller does not value the objects; the buyer values object $k$ by $\theta^k$ ($k = 1, 2$) and getting both objects by $\theta^1 + \theta^2$.

(a) Suppose that valuations are independently distributed and, for $k = 1, 2$,

$$\theta^k = \begin{cases} 10 & \text{with probability } \frac{1}{2} \\ 22 & \text{with probability } \frac{1}{2}. \end{cases}$$

What are the optimal prices and the corresponding revenue if the seller sells the objects separately? What is the optimal price and the corresponding revenue if the seller only sells the bundle?

(b) Suppose that valuations are independently distributed and, for $k = 1, 2$,

$$\theta^k = \begin{cases} 10 & \text{with probability } \frac{1}{2} \\ 50 & \text{with probability } \frac{1}{2}. \end{cases}$$

What are the optimal prices and the corresponding revenue if the seller sells the objects separately? What is the optimal price and the corresponding revenue if the seller only sells the bundle?

(c) Suppose the seller sets a price for each object and a price for the bundle of both objects. Determine the optimal prices if valuations are identically, independently, and uniformly distributed on $[0, 1]$.

(d) Suppose valuations are independently distributed and, for $k = 1, 2$,

$$\theta^k = \begin{cases} 1 & \text{with probability } \frac{1}{6} \\ 2 & \text{with probability } \frac{1}{2} \\ 4 & \text{with probability } \frac{1}{3} \end{cases}$$

The expected revenue in the optimal deterministic mechanism is $\frac{29}{9}$.

Suppose the seller offers the following menu: A lottery which yields with probability $\frac{1}{2}$ object 1 and nothing otherwise, a lottery which yields with probability $\frac{1}{2}$ object 2 and nothing otherwise, and getting the bundle of both objects for sure. Show that the seller can obtain a larger expected revenue offering this menu compared to the optimal deterministic mechanism.

4. *Interdependent value auction*

Suppose there is one object for sale and $N$ potential buyers. Each agent privately observes a signal $X_i$, which is independently and identically distributed on $[0, \overline{X}]$ with cdf $F$ and density $f$. Denote by $G$ the cdf of the first-order statistic of $N - 1$ of these random variables.

Buyers have quasi-linear utilities: in case of winning the object, buyer $i$ gets utility $v(x_i, x_{-i}) - p$, where $p$ denotes the payment made, and he gets utility of 0 in case of not winning. Suppose that $v$ is positive, strictly increasing in all signals, symmetric in the last $N-1$ signals, and denote by $\overline{v}(x_i, y)$ the expected valuation of agent $i$ given he received signal $x_i$ and the highest signal among all other signals has value $y$.

(a) Show: In a second price auction, each agent bidding according to the bid function $\beta(x_i) = \overline{v}(x_i, x_i)$ is a Bayes-Nash equilibrium.

Is it a dominant strategy to follow this bid function? Is it an ex-post equilibrium?

(b) Consider an open English auction. A symmetric strategy in an English auction is a collection $\boldsymbol{\beta} = (\beta^N, \beta^{N-1}, ..., \beta^2)$ of $N-1$ functions $\beta^k : [0, \overline{X}] \times \mathbb{R}_+^{N-k} \to \mathbb{R}_+$. The interpretation is that $\beta^k(x, p_{k+1}, ..., p_N)$ is the price at which bidder 1 will drop out of the auction if the number of bidders who are still active is $k$, his own signal is $x$, and the prices at which the other $N-k$ bidders dropped out were $p_{k+1} \geq p_{k+2} \geq ... \geq p_N$.

Describe a symmetric Bayes-Nash equilibrium of the open English auction and show that this strategy profile constitutes indeed an equilibrium.

Is it an equilibrium in dominant strategies? Is it an ex-post equilibrium?

(c) Show that the symmetric bidding strategies $\beta(x) = \frac{1}{G(x)} \int_0^x v(y, y) dG(y)$ form a Bayes-Nash equilibrium of the first-price auction.

(d) Suppose $N = 2$, bidder $i$'s valuation is $v_i(x_i, x_j) = \eta x_i + (1 - \eta) x_j$. For which $\eta$ is the outcome of the second-price auction efficient?