Microeconomics II (PhD)
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Part A

You can use the following envelope theorem in the next exercise.

Theorem (Milgrom and Segal, 2002).
Let X be an arbitrary set, $T = [\underline{t}, \overline{t}]$,[1] and $f : X \times T \to \mathbb{R}$. Denote

$$V(t) = \sup_{x \in X} f(x,t) \tag{1}$$

$$X^*(t) = \{x \in X | f(x,t) = V(t)\}. \tag{2}$$

Suppose that $f(x, \cdot)$ is differentiable for all $x \in X$, $f_t(x, \cdot)$ is uniformly bounded and that $X^*(t) \neq \emptyset$ for almost all t. Then for any selection $x^*(t) \in X^*(t)$,

$$V(t) = V(\underline{t}) + \int_{\underline{t}}^{t} f_t(x^*(s), s) ds. \tag{3}$$

1. Consider the general mechanism design setting from the lecture, where $v_i(k, \theta_i)$ denotes the value of allocation k to agent i with type θ_i. Suppose that $\Theta_i = [\underline{\theta}_i, \overline{\theta}_i] \subset \mathbb{R}$ and that v_i is differentiable in θ_i for all k and the derivative is uniformly bounded. Given a direct revelation mechanism (k, t), let $U_i(\theta) = v_i(k(\theta), \theta_i) + t_i(\theta)$ be the utility of agent i if θ is the profile of types and all agents report truthfully.
   (a) Show that if the direct revelation mechanism (k, t) is implementable in dominant strategies, then

$$U_i(\theta) = U_i(\underline{\theta}_i, \theta_{-i}) + \int_{\underline{\theta}_i}^{\theta_i} \frac{\partial v_i(k(s, \theta_{-i}), s)}{\partial \theta_i} ds. \tag{ICFOC}$$

   Suppose that $v_i(k, \theta_i)$ has the single-crossing property: $\frac{\partial^2 v_i(k,\theta_i)}{\partial k \, \partial \theta_i}$ exists and is stricly positive for all $k \in K$ and $\theta_i \in \Theta_i$.
   (b) Show that if the direct revelation mechanism (k, t) is implementable in dominant strategies, then $k(\theta_i, \theta_{-i})$ is weakly increasing in θ_i for all θ_{-i}.
   (c) Show that any monotone mechanism that satisfies (ICFOC) is implementable in dominant strategies.

Solution: Denote the gains from reporting $\hat{\theta}_i$ instead of the true type θ_i by

$$\ell_i(\theta_i, \hat{\theta}_i, \theta_{-i}) := v_i(k(\hat{\theta}_i, \theta_{-i}), \theta_i) + t_i(\hat{\theta}_i, \theta_{-i}) - U_i(\theta_i, \theta_{-i}).$$

Then:

$$\begin{aligned} \frac{\partial \ell_i(\theta_i, \hat{\theta}_i, \theta_{-i})}{\partial \theta_i} &= \frac{\partial v_i(k(\hat{\theta}_i, \theta_{-i}), \theta_i)}{\partial \theta_i} - \frac{v_i(k(\theta_i, \theta_{-i}), \theta_i)}{\partial \theta_i} \\ &= \int_{k(\theta_i, \theta_{-i})}^{k(\hat{\theta}_i, \theta_{-i})} \frac{\partial^2 v_i(k, \theta_i)}{\partial \theta_i \, \partial k} dk, \end{aligned}$$

where the first line uses (ICFOC) and the second line follows from the fundamental theorem of calculus (assuming, for example, that the cross-derivative is a continuous function).
Because the cross-derivative is greater than 0 and k is weakly increasing, the integral is ≥ 0 if and only if $\hat{\theta}_i \geq \theta_i$. Hence, $\ell_i(\theta_i, \hat{\theta}_i, \theta_{-i})$ is increasing in θ_i for $\theta_i \leq \hat{\theta}_i$ and decreasing for $\theta_i \geq \hat{\theta}_i$. Therefore, for every $\hat{\theta}_i$, it is maximized for $\theta_i = \hat{\theta}_i$, and the maximum is $\ell_i(\hat{\theta}_i, \hat{\theta}_i, \theta_{-i}) = 0$. Consequently, the gains from lying are weakly negative for all types and the mechanism is DIC.

[1] This result holds more generally, for example if $T \subset \mathbb{R}^n$ is convex.

(d) Discuss the relation of these results to the result that you saw in the lecture.

(e) Show: If a direct revelation mechanism implements the value-maximizing allocation rule in dominant strategies, then it is a VCG mechanism.

Solution: The VCG mechanism with $h_i(\theta_{-i}) \equiv 0$ implements the value-maximizing allocation rule k^* in dominant strategies. Therefore,

$$t_i^{VCG}(\theta_i, \theta_{-i}) = \sum_{j \neq i} v_j(k^*(\theta), \theta_j) = t_i^{VCG}(\underline{\theta_i}, \theta_{-i}) + \int_{\underline{\theta}_i}^{\theta_i} \frac{\partial v_i(k^*(s, \theta_{-i}), s)}{\partial \theta_i} ds.$$

Moreover, for any mechanism (k^*, t) that is DIC we have

$$t(\theta_i, \theta_{-i}) = t(\underline{\theta}_i, \theta_{-i}) + \int_{\underline{\theta}_i}^{\theta_i} \frac{\partial v_i(k^*(s, \theta_{-i}), s)}{\partial \theta_i} ds.$$

Rearranging, we get

$$t(\theta_i, \theta_{-i}) = \sum_{j \neq i} v_j(k^*(\theta), \theta_j) + t(\underline{\theta}_i, \theta_{-i}) - t_i^{VCG}(\underline{\theta}_i, \theta_{-i})$$

and hence (k^*, t) is a VCG mechanism as well.

Part B

2. Suppose there is one agent, three potential types $(\theta^1, \theta^2, \theta^3)$ and three alternatives (a, b, c). The valuation the agent has for an alternative given his type is given by the following matrix:

	θ^1	θ^2	θ^3
a	0	-1	x
b	1	0	-1
c	-1	1	0

Consider the function k' such that $k'(\theta^1) = a$, $k'(\theta^2) = b$, and $k'(\theta^3) = c$.

(a)

Definition 1. *A decision rule k is* weakly monotone *if for all θ^i, θ^j,*

$$v(k(\theta^i), \theta^i) - v(k(\theta^j), \theta^i) \geq v(k(\theta^i), \theta^j) - v(k(\theta^j), \theta^j).$$

Suppose $x = 1$. Is k' weakly monotone? Is it implementable in the sense that there is a payment rule t such that (k', t) is incentive compatible? How does this relate to the result you saw in the lecture?

(b)

Definition 2. *A decision rule k is* cyclically monotone *if for every sequence of types of length $l \in \mathbb{N}$, $(\theta^1, \theta^2, ..., \theta^l)$, with $\theta^l = \theta^1$, we have*

$$\sum_{\kappa=1}^{l-1} v(k(\theta^\kappa), \theta^{\kappa+1}) - v(k(\theta^\kappa), \theta^\kappa) \leq 0.$$

Show that every implementable decision rule k is cyclically monotone.

(c) For which values of x is k' cyclically monotone?

3. There is one seller with two objects, and one buyer. The seller does not value the objects; the buyer values object k by θ^k ($k = 1, 2$) and getting both objects by $\theta^1 + \theta^2$.

   (a) Suppose that valuations are independently distributed and, for $k = 1, 2$,

$$\theta^k = \begin{cases} 10 & \text{with probability } \frac{1}{2} \\ 22 & \text{with probability } \frac{1}{2}. \end{cases}$$

   What are the optimal prices and the corresponding revenue if the seller sells the objects separately? What is the optimal price and the corresponding revenue if the seller only sells the bundle?

   (b) Suppose that valuations are independently distributed and, for $k = 1, 2$,

$$\theta^k = \begin{cases} 10 & \text{with probability } \frac{1}{2} \\ 50 & \text{with probability } \frac{1}{2}. \end{cases}$$

   What are the optimal prices and the corresponding revenue if the seller sells the objects separately? What is the optimal price and the corresponding revenue if the seller only sells the bundle?

   (c) Suppose the seller sets a price for each object and a price for the bundle of both objects. Determine the optimal prices if valuations are identically, independently, and uniformly distributed on $[0, 1]$.

   (d) Suppose valuations are independently distributed and, for $k = 1, 2$,

$$\theta^k = \begin{cases} 1 & \text{with probability } \frac{1}{6} \\ 2 & \text{with probability } \frac{1}{2} \\ 4 & \text{with probability } \frac{1}{3} \end{cases}$$

   The expected revenue in the optimal deterministic mechanism is $\frac{29}{9}$.
   Suppose the seller offers the following menu: A lottery which yields with probability $\frac{1}{2}$ object 1 and nothing otherwise, a lottery which yields with probability $\frac{1}{2}$ object 2 and nothing otherwise, and getting the bundle of both objects for sure. Show that the seller can obtain a larger expected revenue offering this menu compared to the optimal deterministic mechanism.

4. *Interdependent value auction*

   Suppose there is one object for sale and N potential buyers. Each agent privately observes a signal X_i, which is independently and identically distributed on $[0, \overline{X}]$ with cdf F and density f. Denote by G the cdf of the first-order statistic of $N - 1$ of these random variables.

   Buyers have quasi-linear utilities: in case of winning the object, buyer i gets utility $v(x_i, x_{-i}) - p$, where p denotes the payment made, and he gets utility of 0 in case of not winning. Suppose that v is positive, strictly increasing in all signals, symmetric in the last $N - 1$ signals, and denote by $\overline{v}(x_i, y)$ the expected valuation of agent i given he received signal x_i and the highest signal among all other signals has value y.

   (a) Show: In a second price auction, each agent bidding according to the bid function $\beta(x_i) = \overline{v}(x_i, x_i)$ is a Bayes-Nash equilibrium.
   Is it a dominant strategy to follow this bid function? Is it an ex-post equilibrium?

   Solution: Fix a bidder i, suppose this bidder bids b and the highest signal among all other bidders is y. Bidder i will win the auction if the higest bid among all others is below b, $\beta(y) \leq b$, which is equivalent to $y \leq \beta^{-1}(b)$. Hence, the expected payoff of bidder i with signal x who bids b, given the others use the proposed bidding strategies, is

$$\Pi(b, x) := \int_0^{\beta^{-1}(b)} \bar{v}(x, y) - \bar{v}(y, y) \; dG(y). \tag{4}$$

Since $\bar{v}$ is strictly increasing, the integrand is > 0 for all $y < x$ and < 0 for all $y > x$. Therefore, Π is maximized for a bid b such that

$$\bar{v}(x, \beta^{-1}(b)) - \bar{v}(\beta^{-1}(b), \beta^{-1}(b)) = 0,$$

which yields $b = \beta(x)$. □

If $n = 2$, $\bar{v}$ and the argument above is completely independent of the distribution G and hence the equilibrium is actually an ex-post equilibrium. For $n > 2$, $\bar{v}$ depends on G and one can easily construct an example where an agent regrets his bid ex-post (see your notes from the tutorial for an example).

Clearly, the equilibrium is not in dominant strategies (even is $n = 2$).

(b) Consider an open English auction. A symmetric strategy in an English auction is a collection $\boldsymbol{\beta} = (\beta^N, \beta^{N-1}, ..., \beta^2)$ of $N - 1$ functions $\beta^k : [0, \overline{X}] \times \mathbb{R}_+^{N-k} \to \mathbb{R}_+$. The interpretation is that $\beta^k(x, p_{k+1}, ..., p_N)$ is the price at which bidder 1 will drop out of the auction if the number of bidders who are still active is k, his own signal is x, and the prices at which the other $N - k$ bidders dropped out were $p_{k+1} \geq p_{k+2} \geq ... \geq p_N$.

Describe a symmetric Bayes-Nash equilibrium of the open English auction and show that this strategy profile constitutes indeed an equilibrium.

Is it an equilibrium in dominant strategies? Is it an ex-post equilibrium?

Solution: Let p^N denote the price at which the first bidder drops out, and define $x^N = \beta^{N-1}(p^N)$.

$$\beta^N(x_i) = v(x_i, x_i, ...) \tag{5}$$

$$\beta^{N-1}(x_i, p^N) = v(x_i, x_i, ..., x_N)$$

$$\beta^k(x_i, p^{k+1}, ..., p^N) = v(\underbrace{x_i, ..., x_i}_{k \text{ times}}, x^{k+1}, ..., x^N), \tag{6}$$

where x^{k+1} is defined implicitly by $p^{k+1} = \beta^{k+1}(x^{k+1}, p^{k+2}, ..., p^N)$.

Claim: The bidding strategies defined above form an ex-post equilibrium in the open English auction.

Proof. Fix an arbitrary signal realization, suppose all others follow this strategy and focus on bidder 1.

Case (i): Bidder 1 gets the object when following this strategy.

Payoff: $v(x_1, y_1, ..., y_{N-1}) - v(y_1, y_1, ..., y_{N-1})$, where y_k denotes the $k-$largest of $\{x_2, ..., x_N\}$. Since strategies are symmetric, $x_1 \geq y_1$. Hence, payoff is weakly positive. There is no profitable deviation: bidding lower does not change the payoff or leads to payoff 0. Bidding higher gives the same payoff.

Case (ii): Bidder 1 does not get the object when following this strategy.

Note that $x_i \leq y_1$. Any deviation that leads bidder 1 to win will give him a payoff $v(x_1, y_1, ..., y_{N-1}) - v(y_1, y_1, ..., y_{N-1}) \leq 0$, and all other deviations give him payoff 0. By using the proposed strategy, he gets payoff 0, hence there is no profitable deviation. □

Clearly, the strategies are not dominant.

(c) Show that the symmetric bidding strategies $\beta(x) = \frac{1}{G(x)} \int_0^x v(y, y) dG(y)$ form a Bayes-Nash equilibrium of the first-price auction.

Solution: Note that β is strictly increasing and every bidder uses the same strategy. The expected payoff to a bidder with signal x who bids $\beta(z)$ is therefore

$$\Pi(x, z) := \int_0^z \bar{v}(x, y) - \beta(z) dG(y) = \int_0^z \bar{v}(x, y) - \bar{v}(y, y) \; dG(y).$$

Hence, $\Pi(x, x) - \Pi(x, z) = \int_z^x \bar{v}(x, y) - \bar{v}(y, y) dG(y) \geq 0$ for all z. Therefore there is no profitable deviation in the range of β. Clearly, no bid outside this range is profitable, and hence the strategies form a Bayesian equilibrium.

(d) Suppose $N = 2$, bidder i's valuation is $v_i(x_i, x_j) = \eta x_i + (1 - \eta) x_j$. For which η is the outcome of the second-price auction efficient?

Solution: Because the bidding strategies are strictly increasing, the bidder with the higher signal receives the object.

Efficiency therefore requires that $\bar{v}(x, y) \geq \bar{v}(y, x)$ for all $x > y$. Rearranging, this is equivalent to $\eta \geq \frac{1}{2}$.