Microeconomics II (PhD)
Tutorial 2, May 2

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Part A

1. Solve Exercises 23.C.3 and 23.C.4 in MWG.

2. Solve Exercise 23.AA.1 in MWG.

3. Consider a bilateral trade setting, where the seller values the object at 0 and the buyer is privately informed about his valuation v. There are two periods, and both agents have a common discount rate $0 < \delta \leq 1$. In each period, the seller posts a price p_t and the buyer decides whether to buy at the proposed price or whether to reject the current offer.

   (a) Assume that $v \sim U[0,1]$ and suppose the seller can commit to a price schedule (p_1, p_2) at the start of period 1. Which prices does he propose?

   Assume for the remainder of the exercise that the seller cannot commit to the second-period price when proposing the first-period price.

   (b) Assume that $v \sim U[0,1]$ and suppose that $\delta = 1$. Find a perfect Bayesian equilibrium of the game. Is it an equilibrium if $\delta < 1$?

   (c) Suppose now that $\delta < 1$, and that $v = v_H$ with probability μ_H and $v = v_L$ with probability $1 - \mu_H$, where $v_H > v_L > 0$. Assume that $\mu_H < \frac{v_L}{v_H}$. Find the perfect Bayesian equilibrium.

   (d) *(verbally)* Suppose now that the seller can propose in each period a general mechanism. Is it without loss of generality for the seller to propose in each period a direct and incentive-compatible mechanism (that is, does the revelation principle apply in this situation)?

Part B

4. Consider the quasi-linear private values environment from the lecture. Let $\alpha_1, ..., \alpha_I \in \mathbb{R}_+ \setminus \{0\}$ and $\lambda_1, ..., \lambda_K \in \mathbb{R}$. A function $k : \Theta \to K$ is called an *affine maximizer* if

$$k(\theta) \in \arg\max_k \sum_{i=1}^{I} \alpha_i \cdot v_i(k, \theta_i) + \lambda_k.$$

   Show: $k : \Theta \to K$ is truthfully implementable in dominant strategies if k is an affine maximizer.

5. Solve Exercise 23.C.10 in MWG.

   Assume throughout the exercise that (23.C.8) is a necessary condition for $(k^*, t_1, ..., t_I)$ to be truthfully implementable in dominant strategies. In part c insert "implementable" before "ex post efficient social choice function" and suppose that $V_i(\theta_{-i})$ is I times continuously differentiable for each i.

6. Suppose there are two agents and the question whether a brigde should be built. The net valuation of agent i for having a bridge is θ_i. Utilities are quasi-linear: agent i gets utility $\theta_i + t_i$ if the bridge is built and t_i otherwise, where t_i denotes the transfer he receives.

   Assume that $\Theta_i = \mathbb{R}$ for $i = 1, 2$.

   (a) Show that there exists no VCG mechanism such that $\sum_i t_i(\theta) = 0$ for all $\theta \in \mathbb{R}^n$.

   Hint: Consider distinct types $\theta_1,\ \theta_1',\ \theta_2,\ \theta_2'$ such that

$$\theta_1 + \theta_2 > 0,\ \theta_1' + \theta_2 < 0,\ \theta_1 + \theta_2' > 0,\ \theta_1' + \theta_2' < 0.$$

Solution: As proposed in the hint, choose distinct $\theta_1, \theta_1', \theta_2, \theta_2'$ such that

$$\theta_1 + \theta_2 > 0,\ \theta_1' + \theta_2 < 0,\ \theta_1 + \theta_2' > 0,\ \theta_1' + \theta_2' < 0.$$

This is possible because $\Theta_i = \mathbb{R}$. Denoting by q the probability that the bridge will be built, an efficient allocation rule has

$$q(\theta_1, \theta_2) = q(\theta_1, \theta_2') = 1 \text{ and } q(\theta_1', \theta_2) = q(\theta_1', \theta_2') = 0.$$

To obtain a contradiction, suppose there exists a budget-balanced VCG mechanism. This implies

$$\begin{aligned} \theta_2 + h_1(\theta_2) + \theta_1 + h_2(\theta_1) &= 0 \\ h_1(\theta_2) + h_2(\theta_1') &= 0 \\ \theta_2' + h_1(\theta_2') + \theta_1 + h_2(\theta_1) &= 0 \\ h_1(\theta_2') + h_2(\theta_1') &= 0. \end{aligned}$$

The second and fourth line imply $h_1(\theta_2) = h_1(\theta_2')$. Plugging this into the first line and substracting the third line yields $\theta_2 = \theta_2'$, which contradicts the choice of the θ's.

(b) *(verbally)* The above result extends to n agents. What can you conclude from this for general private value settings (that is, not only binary public good settings) if all valuations are possible, i.e. $\{v_i(\cdot, \theta_i) | \theta_i \in \Theta_i\} = \mathcal{V}$?

Solution: Because there is no budget-balanced VCG mechanism in the binary alternatives setting, there cannot be one in more general settings with several alternatives.

7. Consider again the public good setting from the previous exercise. Suppose now that the net valuation of agent i for having a bridge, θ_i, is independently and uniformly distributed on [-3,3].

(a) Assume agents can either vote in favor or against the bridge and there are no transfers. The bridge will be built if and only if both agents vote for it. What is an equilibrium in dominant strategies? If agents follow these strategies, what is the expected aggregate utility (that is, the sum of the agents expected utilities)?

Solution: It is a weakly dominant strategy to vote for the bridge if and only if your valuation for the bridge is strictly positive. If agents follow this strategy, the bridge will be built with probability $\frac{1}{4}$. The expected valuation of agent i given that he prefers the bridge is $\mathbb{E}[\theta_i | \theta_i > 0] = \frac{3}{2}$. Therefore, expected utilitarian welfare is $W_{voting} = \frac{1}{4} \cdot \sum_i \frac{3}{2} = \frac{3}{4}$.

(b) Suppose that agents' valuations were observed by a utilitarian social planner. Which decision rule should he implement and what is the resulting expected aggregate utility?

Solution: The planner should build the bridge whenever the sum of valuations is positive:

$$q^*(\theta_1, \theta_2) = \begin{cases} 1 \text{ if } \theta_1 + \theta_2 > 0 \\ 0 \text{ else.} \end{cases}$$

To obtain the corresponding welfare, one computes

$$W_{FB} = \frac{1}{2} \mathbb{E}[\theta_1 + \theta_2 | \theta_1 + \theta_2 > 0] = 1.$$

One can either do the computation explicitly (using integrals), use the distribution function of the sum of two uniform variables (which is the convolution of the uniform distribution), or use a geometric argument (the above expected value corresponds to the geometric center of a triangle, for which there are simple formulas).

(c) Assume that transfers are feasible. What is the expected aggregate utility if the Pivotal mechanism is implemented?

Solution: The allocation rule is the same as in part b, therefore we only add (negative) transfers:

$$W_{VCG} = W_{FB} + \mathbb{E}[t_1(\theta_1, \theta_2) + t_2(\theta_1, \theta_2)] = 2\ W_{FB} + \mathbb{E}[h_1(\theta_2) + h_2(\theta_1)]. \tag{1}$$

Note that in the pivotal mechanism

$$h_i(\theta_{-i}) = \begin{cases} -\theta_{-i} & \text{if } \theta_{-i} \geq 0 \\ 0 & \text{otherwise,} \end{cases} \tag{2}$$

and consequently $\mathbb{E}[h_1(\theta_2)] = -\frac{3}{4}$. This yields $W_{VCG} = \frac{1}{2}$.