Mechanism Design and Social Choice
Part III: Social Choice Theory

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(revised version)

1 Introduction

1.1 Overview

- Social choice theory: analysis of collective decision making

- A group of individuals has to make a decision (electing representatives, deciding laws, allocating scarce resources,...).

  How can we **aggregate** the diverse **individual opinions** into one consistent **social ranking** of alternatives, or more directly into a **social decision**?
  - Examples: Voting, dictatorship, lottery, auction,...
  - Does aggregation procedure fulfill some fundamental desirable properties?
  - If individual opinions are private information, will they be reported truthfully?

- Simple and very general formal structure
  - compared to Parts I and II, only very few assumptions on preferences and set of alternatives.
  - only ordinal preferences, no information on intensity of preferences is used, no comparison of utility across individuals (no money)

Literature

Main textbook references:

- Mas-Colell, A., Whinston, M.D., and Green, J.R. (1995). *Microeconomic Theory*, Oxford University Press. → Chapter 21
- Jehle, G.A. and Reny, P.J. (2011). *Advanced Microeconomic Theory*, Financial Times Prentice Hall. → Chapter 6
- Gaertner, W. (2009). *A Primer in Social Choice Theory*, Oxford University Press. → Chapters 1–3, 5

More advanced/comprehensive:

- Moulin, H. (1988). *Axioms of Cooperative Decision Making*, Cambridge University Press.
- Austen-Smith, D., and Banks, J. S. (2000 & 2005). *Positive Political Theory I & II*, University of Michigan Press.

Outline

- Social Welfare Functions
  - Arrow's impossibility result
  - Restricted environments (May's Theorem, Median Voter Theorem)
- Social Choice Functions
  - The Gibbard-Satterthwaite Theorem
  - Restricted environments

1.2 Preliminaries

Environment:

- society of $n \geq 2$ agents $N := \{1, \ldots, n\}$
- finite set X of mutually exclusive alternatives
- each agent i has a **rational preference relation** R_i defined on X

Aim: Aggregate a profile of individual preferences $(R_1, \ldots, R_n)$ into

1. one **social preference relation** R (*social welfare function*)
2. a **social choice** from X (*social choice function*)

Preference relations

A **preference relation** R_i is a binary relation on X.

A binary relation on X is a collection of ordered pairs $(x, y) \in X \times X$,
i.e., R_i is a subset of $X \times X$.

Notation:

- $x \; R_i \; y$ denotes $(x, y) \in R_i$, meaning x *is weakly preferred to* y.
- $x \; P_i \; y$ denotes $x \; R_i \; y$ but not $y \; R_i \; x$, meaning x *is strictly preferred to* y.
- $x \; I_i \; y$ denotes $x \; R_i \; y$ and $y \; R_i \; x$, meaning x *is indifferent to* y.

(alternative notation: $\succeq_i$, $\succ_i$, and $\sim_i$)

A preference relation R_i is **rational** if it is complete and transitive.

- R_i is **complete** if for all $x, y \in X$ either $x \; R_i \; y$ or $y \; R_i \; x$ (or both).
- R_i is **transitive** if for all $x, y, z \in X$, $x \; R_i \; y$ and $y \; R_i \; z$ imply $x \; R_i \; z$.

Domain of preferences

Let $\mathscr{R}$ denote the set of all rational preference relations R_i on X.

Restricted domains:

- Let $\mathscr{P} \subset \mathscr{R}$ denote the set of all strict preference relations, i.e., all $R_i \in \mathscr{R}$ such that for all $x, y \in X$ either $x \ P_i \ y$ or $y \ P_i \ x$.
- single-peaked preferences (introduced later)
- Part I: each agent indifferent between alternatives (matchings) where she obtains the same partner/object; strict preference if partner/object differs
- Part II: quasi-linear utilities

2 Social Welfare Functions

2.1 Social welfare functions

Definition

A **social welfare function (SWF)** on domain $\mathscr{A} \subseteq \mathscr{R}^n$ is a mapping $F\colon \mathscr{A} \to \mathscr{R}$ that assigns a rational social preference relation $F(R_1, \ldots, R_n)$ to any profile of individual preferences $(R_1, \ldots, R_n) \in \mathscr{A}$.

- A social welfare function on $\mathscr{A}$ is a *consistent* procedure for aggregating individual preferences in $\mathscr{A}$ into a social preference relation.
- The resulting social preference relation is required to be **rational**, i.e., **complete** and **transitive**.

Notation: For $R = F(R_1, \ldots, R_n)$,

- $x \; R \; y$ means that x is *weakly socially preferred* to y;
- P and I denote *strict social preference* and *social indifference*.

Examples I: Pairwise majority voting

- **Simple Majority Rule**, $R = F^{SM}(R_1, \dots, R_n)$:

$$x \; R \; y \quad \text{if and only if} \quad \big|\{i \in N : x \; P_i \; y\}\big| \geq \big|\{i \in N : y \; P_i \; x\}\big|.$$

- **Absolute Majority Rule**, $R = F^{AM}(R_1, \dots, R_n)$:

$$x \; R \; y \quad \text{if and only if} \quad \big|\{i \in N : x \; R_i \; y\}\big| \geq \frac{n}{2}.$$

Example: In the Bundestag, many decisions are made by simple majority; absolute majority is necessary, e.g., for appointing the chancellor.

Note: If $R_i \in \mathscr{P}$ (*strict* preference relation) for all $i \in N$, then $F^{SM} = F^{AM}$.

Condorcet paradox in majority voting

Let $N = \{1, 2, 3\}$ and $X = \{x, y, z\}$.
Consider the following (strict) preference relations:

R_1	R_2	R_3
x	y	z
y	z	x
z	x	y

- Pairwise majority voting yields $x\ P\ y$, $y\ P\ z$, and $z\ P\ x$.
- Resulting social preferences not transitive!
  $\Rightarrow$ Pairwise majority voting is not a SWF on domain $\mathscr{R}^n$.

Examples II: Scoring rules

Suppose $|X| = m$ and each R_i is a *strict* preference relation.

Let $s = (s_1, s_2, \ldots, s_m)$ be a non-increasing sequence of real numbers with $s_1 > s_m$. For each $i \in N$ and $x \in X$, set $S_i(x) = s_k$ if x is agent i's kth most preferred alternative according to R_i.

- **Scoring rule**, $R = F^S(R_1, \ldots, R_n)$ induced by s:

$$x \; R \; y \quad \text{if and only if} \quad \sum_{i \in N} S_i(x) \geq \sum_{i \in N} S_i(y).$$

- Special case: **Borda count** if $s_k = m - k + 1$ for all $k \in \{1, \ldots, m\}$.

Examples: Eurovision song contest, sports rankings

Examples III: Dictatorship

Definition

A social welfare function F on $\mathscr{A}$ is **dictatorial** if for some agent $i \in N$, $x \; P_i \; y$ implies $x \; P \; y$ for all $(R_1, \ldots, R_n) \in \mathscr{A}$ and all $x, y \in X$.

Note:

- An agent is a dictator only if she dictates strict preferences in every possible social decision problem.
- If individual indifferences are possible, dictatorship does not fully pin down the SWF.

Preference aggregation: The axiomatic method

- While individuals will rarely agree on the best alternative, they may agree on some basic properties that the social decision process should satisfy.

- *The axiomatic method:*
  - Propose a set of normatively appealing properties, or *axioms*, that any aggregation procedure *should* satisfy.
  - *Characterize* the class of aggregation procedures that satisfy these properties (or show that they are incompatible).

2.2 Arrow's impossibility result

Arrow (1951, 1963) considers SWFs on domain $\mathscr{R}^n$ and defines three desirable properties for a SWF: *weakly Paretian*, *non-dictatorial*, and *independence of irrelevant alternatives*.

Definition

A social welfare function F is **weakly Paretian (WP)** if

$$x \; P \; y \quad \text{whenever} \quad x \; P_i \; y \text{ for all } i \in N.$$

Definition

A social welfare function F is **non-dictatorial (ND)** if it is not dictatorial.

Independence of irrelevant alternatives

Definition

Let $R = F(R_1, \ldots, R_n)$ and $\tilde{R} = F(\tilde{R}_1, \ldots, \tilde{R}_n)$.
A social welfare function F satisfies **independence of irrelevant alternatives (IIA)** if whenever $(R_1, \ldots, R_n), (\tilde{R}_1, \ldots, \tilde{R}_n)$ and $x, y \in X$ are such that

$$x \; R_i \; y \Leftrightarrow x \; \tilde{R}_i \; y \quad \text{and} \quad y \; R_i \; x \Leftrightarrow y \; \tilde{R}_i \; x \quad \text{for all } i \in N,$$

then

$$x \; R \; y \Leftrightarrow x \; \tilde{R} \; y \quad \text{and} \quad y \; R \; x \Leftrightarrow y \; \tilde{R} \; x.$$

- IIA requires that if for two profiles of preferences every agent has exactly the same ranking of x and y, then the social ranking of x and y must also be the same for the two profiles.

  $\rightarrow$ The social ranking of x and y should depend only on the individual rankings of x and y (and not on individual rankings of other alternatives).

Examples

- Simple and absolute majority rules satisfy WP, IIA, and ND but may not produce a transitive relation (*Condorcet paradox*).
- Scoring rules with $s_1 > s_2 > \ldots > s_m$ satisfy WP and ND but not IIA. ($\rightarrow$ Exercise)
- Consider a dictatorship where the dictator also dictates social indifference, i.e., there is an i such that $F(R_1, \ldots, R_n) = R_i$ for all $(R_1, \ldots, R_n) \in \mathscr{A}$. Such a SWF F on $\mathscr{A}$ satisfies WP and IIA.
- A SWF that ignores individual preferences and always produces the same social preference relation satisfies IIA and ND but not WP.

$\Longrightarrow$ WP, IIA, and ND are independent.

Arrow's Theorem

Theorem (Arrow, 1951, 1963)

Suppose $|X| \geq 3$. If a SWF on domain $\mathscr{R}^n$ satisfies WP and IIA, then it is dictatorial.

- Arrow's theorem shows that WP and IIA *characterize* dictatorship.
- If we view WP, IIA, and ND as fundamental axioms that any SWF should satisfy, Arrow's theorem is an **impossibility result**:

  There is no SWF on $\mathscr{R}^n$ which satisfies all three properties simultaneously!

Proof

See also Mas-Colell et al. (1995, Subsect. 21.C) or Gaertner (2009, Subsect. 2.2).

Consider a SWF on $\mathscr{R}^n$ that satisfies WP and IIA.

Definition

A set of agents V is **almost decisive for** (x, y) if

$$x \; P_i \; y \text{ for all } i \in V \text{ and } y \; P_j \; x \text{ for all } j \notin V \implies x \; P \; y.$$

A set of agents V is **decisive for** (x, y) if

$$x \; P_i \; y \text{ for all } i \in V \implies x \; P \; y.$$

Note: decisive for $(x, y) \implies$ almost decisive for (x, y)

We prove the theorem by showing the following:

1. There is an agent who is almost decisive for some (x, y).
2. If an agent is almost decisive for some (x, y), then this agent is a dictator.

Lemma

There is an agent who is almost decisive for some (x, y).

Proof.

Suppose, by *contradiction*, that no single agent is almost decisive.

- An almost decisive set exists as N is almost decisive because of WP. Let V be one of the **smallest almost decisive sets**; note that $|V| \geq 2$.
- Let V be almost decisive for (x, y) and consider the preference profile

$$x\ P_i\ y \text{ and } y\ P_i\ z \text{ for one } i \in V,$$
$$z\ P_j\ x \text{ and } x\ P_j\ y \text{ for all } j \in V \setminus \{i\},$$
$$y\ P_k\ z \text{ and } z\ P_k\ x \text{ for all } k \notin V.$$

  Since V is almost decisive for (x, y), $x\ P\ y$. Two possibilities for x vs. z:

  1. $x\ P\ z$: Note that $x\ P_i\ z$ but $z\ P_l\ x$ for all $l \neq i$. Because of IIA, agent i is hence almost decisive for (x, z). $\Longrightarrow$ Contradiction.
  2. $z\ R\ x$: By transitivity, $z\ P\ y$. Note that $z\ P_j\ y$ for $j \in V \setminus \{i\}$ but all other agents strictly prefer y to z. Because of IIA, the set $V \setminus \{i\}$ is hence almost decisive for (z, y). $\Longrightarrow$ Contradiction to V being smallest. □

Lemma

If an agent $k \in N$ is almost decisive for some (x, y), then k is a dictator.

Proof.

1. *If k is almost decisive for (x, y), then for all $z \neq x$, k is decisive for (x, z).*

   Consider the profile $x \; P_k \; y$, $y \; P_k \; z$ and $y \; P_i \; x$, $y \; P_i \; z$ for all $i \neq k$.
   As k is almost decisive for (x, y), $x \; P \; y$. WP implies $y \; P \; z$. By transitivity, $x \; P \; z$.
   Note that $x \; P_k \; z$ whereas for all other agents the preference between x and z was not specified. By IIA, k is hence decisive for (x, z).

2. *If k is almost decisive for (x, y), then for all $z \neq y$, k is decisive for (z, y).*

   Proof similar to ①, using profile $z \; P_k \; x$, $x \; P_k \; y$ and $z \; P_i \; x$, $y \; P_i \; x$ for all $i \neq k$.

3. *For all $u, v \in X$ with $u \neq v$, k is decisive for (u, v).*

   If $v \neq x$, almost decisive for $(x, y) \overset{①}{\Longrightarrow}$ decisive for $(x, v) \overset{②}{\Longrightarrow}$ decisive for (u, v).
   If $v = x$, consider some $z \notin \{x, y\}$. Then, almost decisive for (x, y) $\overset{②}{\Longrightarrow}$ decisive for (z, y) $\overset{①}{\Longrightarrow}$ decisive for (z, v) $\overset{②}{\Longrightarrow}$ decisive for (u, v).

4. *k is decisive for all ordered pairs in $X \iff k$ is a dictator.* □

Discussion

- According to Arrow's theorem, no *ideal* aggregation method exists.
- Arrow's theorem also holds if we assume strict preferences (SWF on $\mathscr{P}^n$).

Next, we consider restricted environments where impossibility does not hold:

2.3 Restricted set of alternatives such that $|X| = 2$.

2.4 Restricted domain of preferences: single-peaked preferences.

2.3 Social preferences over two alternatives

- Many social decisions are taken over only two alternatives:
  public project vs. status quo, presidential election in a two-party system,...

- Simple majority voting F^{SM} is a SWF on $\mathscr{R}^n$ if $|X| = 2$
  (as transitivity has no bite in this case).
- As we have seen, F^{SM} satisfies WP, IIA, and ND.
- Arrow's theorem only holds for $|X| \geq 3$.

- F^{SM} has even stronger properties than WP, IIA, and ND:
  anonymity, *neutrality*, and *positive responsiveness*.
- We will show that these properties characterize F^{SM} if $|X| = 2$.

Anonymity

- Let $\pi : N \to N$ be a permutation of agents.

Definition

A social welfare function F on $\mathscr{A}$ is **anonymous** if

$$F(R_1, \ldots, R_n) = F(R_{\pi(1)}, \ldots, R_{\pi(n)})$$

for all permutations π and all profiles $(R_1, \ldots, R_n) \in \mathscr{A}$.

- *Anonymity* requires that all individuals should count the same in social decision making.
- Note: *anonymity* implies ND.

Neutrality

- Let $(R_1, \ldots, R_n)$ be a preference profile on X.
- Let $\sigma : X \to X$ be a permutation of alternatives.
- Define the preference profile $(R_1^\sigma, \ldots, R_n^\sigma)$ as

$$\sigma(x)\ R_i^\sigma\ \sigma(y) \text{ if and only if } x\ R_i\ y \quad \text{for each } i \in N.$$

Definition

A social welfare function F on $\mathscr{A}$ is **neutral** if

$$\sigma(x)\ F(R_1^\sigma, \ldots, R_n^\sigma)\ \sigma(y) \iff x\ F(R_1, \ldots, R_n)\ y$$

for all $x, y \in X$, all permutations σ, and all profiles $(R_1, \ldots, R_n) \in \mathscr{A}$.

- *Neutrality* requires that all alternatives are treated equally.

Positive responsiveness

Definition

Let $R = F(R_1, \ldots, R_n)$ and $\tilde{R} = F(\tilde{R}_1, \ldots, \tilde{R}_n)$.

A social welfare function F on $\mathscr{A}$ is **positively responsive** if

$$x \; R \; y \text{ implies } x \; \tilde{P} \; y$$

for all $x, y \in X$ and all profiles $(R_1, \ldots, R_n), (\tilde{R}_1, \ldots, \tilde{R}_n) \in \mathscr{A}$ such that

- $\{i \in N : x \; P_i \; y\} \subseteq \{i \in N : x \; \tilde{P}_i \; y\}$,
- $\{i \in N : y \; P_i \; x\} \supseteq \{i \in N : y \; \tilde{P}_i \; x\}$,
- either $y \; R_k \; x$ and $x \; \tilde{P}_k \; y$ or $y \; P_k \; x$ and $x \; \tilde{I}_k \; y$ for at least one $k \in N$.

- If the individual preference of at least one agent changes in favor of x, then x should remain strictly socially preferred to y if it originally was and become strictly socially preferred to y if originally indifferent.
- Note: *positive responsiveness* together with *neutrality* implies WP.

May's Theorem

- Suppose there are only two alternatives: $X = \{x, y\}$
- Each agent i has one of three rankings: $x \; P_i \; y$, $y \; P_i \; x$, or $x \; I_i \; y$

Theorem (May, 1952)

Suppose $|X| = 2$. Then a SWF on domain $\mathscr{R}^n$ is anonymous, neutral, and positively responsive ***if and only if*** *it is the simple majority rule F^{SM}.*

Proof

It is easily seen that F^{SM} is anonymous, neutral, and positively responsive.
We will prove that there is no other SWF that satisfies the three properties.

Suppose the SWF F is anonymous, neutral, and positively responsive.

1. *Anonymity* $\implies$ F depends only on the number of agents who prefer x, the number of indifferent agents, and the number of agents who prefer y:

$$F(R_1, \ldots, R_n) = G(m_x, m_y)$$

   where $m_x := \left|\{i \in N : x \; P_i \; y\}\right|$ and $m_y := \left|\{i \in N : y \; P_i \; x\}\right|$.

2. *Neutrality* requires $x \; G(k, l) \; y \iff y \; G(l, k) \; x$.
   Hence, if $m_x = m_y$, $x \; G(m_x, m_y) \; y$ **and** $y \; G(m_x, m_y) \; x$, i.e., $x \; I \; y$.

3. Consider $(\tilde{R}_1, \ldots, \tilde{R}_n)$ such that $\tilde{m}_x > \tilde{m}_y$ and suppose, wlog,

$$\begin{aligned} x\ \tilde{P}_i\ y \quad & \text{for } i \leq \tilde{m}_x, \\ x\ \tilde{I}_i\ y \quad & \text{for } \tilde{m}_x < i \leq n - \tilde{m}_y, \\ y\ \tilde{P}_i\ x \quad & \text{for } i > n - \tilde{m}_y. \end{aligned}$$

Consider $(R_1, \ldots, R_n)$ such that

$$\begin{aligned} x\ I_i\ y \quad & \text{for } \tilde{m}_y < i \leq \tilde{m}_x, \\ R_i = \tilde{R}_i \quad & \text{for all other } i. \end{aligned}$$

Note that $m_x = m_y = \tilde{m}_y$ and therefore $x\ I\ y$ by ②.

Because of *positive responsiveness*, $x\ I\ y$ implies $x\ \tilde{P}\ y$.

Hence, if $k > l$, then $x\ G(k,l)\ y$ **but not** $y\ G(k,l)\ x$.

4. *Neutrality* implies that, if $k < l$, then $y\ G(k,l)\ x$ **but not** $x\ G(k,l)\ y$.

Consequently, $x\ G(m_x, m_y)\ y$ if $m_x \geq m_y$ and $y\ G(m_x, m_y)\ x$ if $m_x \leq m_y$, which is the definition of F^{SM}. □

2.4 Single-peaked preferences

- A crucial assumption for Arrow's theorem is that all combinations of rational individual preferences are possible.
- In many applications, preferences have more structure.
  $\rightarrow$ can restrict domain

One of the most prominent domain restriction: **single-peakedness**

- The alternatives can be ordered along a single dimension.
- Each agent has a most desired alternative, i.e., a peak.
- Desirability strictly decreases when moving further away from the peak.

Examples

- Politics: left-right scale, tax rate, spending on education
- Level of a public good (number of buses in the city)
- Locating a facility along a line, setting the room temperature of a building.

Single-peaked individual preferences

Let $>$ be a **linear order** on the set of alternatives X:

- For all $x, y \in X$ with $x \neq y$, either $x > y$ or $y > x$ but not both.
- $>$ is transitive.

Definition

A preference relation $R_i \in \mathscr{R}$ is **single-peaked** with respect to the linear order $>$ if there is an alternative $x_i^* \in X$ such that for all $x, y \in X$,

$$x \; P_i \; y \quad \text{if } x_i^* \leq x < y \text{ or } x_i^* \geq x > y.$$

- x_i^* is agent i's **peak**, i.e., agent i's most preferred alternative.
- If alternative x is between the peak x_i^* and alternative y, then $x \; P_i \; y$.

Single-peaked preference profiles

Definition

A profile of preferences $(R_1, \ldots, R_n) \in \mathscr{R}^n$ is **single-peaked** if there is a linear order $>$ such that **for all** $i \in N$, R_i is single-peaked with respect to $>$.

- Single-peakedness requires a certain degree of homogeneity across agents: Each individual R_i has to be single-peaked with respect to the same ordering of the alternatives.

Definition

Consider a single-peaked profile $(R_1, \ldots, R_n)$. Let x_i^* denote agent i's peak. An agent $h \in N$ is a **median voter** if

$$\left|\{i \in N : x_i^* \geq x_h^*\}\right| \geq \frac{n}{2} \text{ and } \left|\{i \in N : x_i^* \leq x_h^*\}\right| \geq \frac{n}{2}.$$

- At least one median voter always exists.

Black's median voter theorem

Recall the **simple majority rule** F^{SM}:

$$x\ F^{SM}(R_1,\ldots,R_n)\ y \iff \big|\{i \in N : x\ P_i\ y\}\big| \geq \big|\{i \in N : y\ P_i\ x\}\big|.$$

Alternative $w \in X$ is a **Condorcet winner** at $(R_1,\ldots,R_n)$ if

$$w\ F^{SM}(R_1,\ldots,R_n)\ x \quad \text{for all } x \in X.$$

Theorem (Black, 1948)

Suppose $(R_1,\ldots,R_n) \in \mathscr{R}^n$ is single-peaked. Then,

- *the peak x_h^* of each median voter h is a Condorcet winner;*
- *the Condorcet winner is unique if n is odd.*

Under single-peakedness, the social ranking derived from F^{SM} has a maximal element that is weakly preferred to all other alternatives. ($\rightarrow$ *acyclic*)

Proof.

Suppose each R_i is single-peaked with respect to $>$. Let h be a median voter.

We must show that $x_h^* \; F^{SM}(R_1, \ldots, R_n) \; y$ for all $y \in X$.

Suppose y is such that $x_h^* > y$. (A very similar argument works for $y > x_h^*$.)

- Let $M = \{i \in N : x_i^* \geq x_h^*\}$.
- Because h is a median voter, $|M| \geq \frac{n}{2}$ and hence $|N \setminus M| \leq \frac{n}{2}$.
- Because preferences are single-peaked, $x_h^* \; P_i \; y$ for all $i \in M$.
- $\left|\{i \in N : x_h^* \; P_i \; y\}\right| \geq |M| \geq \frac{n}{2} \geq |N \setminus M| \geq \left|\{i \in N : y \; P_i \; x_h^*\}\right|$

  $\implies x_h^* \; F^{SM}(R_1, \ldots, R_n) \; y$.

Now, suppose n is odd.

- Then, $x_h^* = x^*$ for each median voter h.
- In the argument above, $|M| > \frac{n}{2} > |N \setminus M|$.
- $\left|\{i \in N : x^* \; P_i \; y\}\right| > \left|\{i \in N : y \; P_i \; x^*\}\right|$ for all $y \neq x^*$.
- x^* is the unique Condorcet winner. □

Example with an even number of agents

Let $N = \{1, 2, 3, 4\}$ and $X = \{a, b, c\}$.

Single-peaked preference profile:

R_1	R_2	R_3	R_4
c	b	a	a
b	c	b	b
a	a	c	c

- The simple majority rule yields: $a\ I\ b$, $b\ P\ c$, and $a\ I\ c$.

  $\implies$ resulting social preferences not transitive
- Two Condorcet winners: a and b.
- Three median voters: 2, 3, and 4.

A possibility result

Let $\mathscr{S} \subset \mathscr{P}^n$ denote the set of all **strict** and **single-peaked** preference profiles.

Theorem

Suppose n *is odd. Then the simple majority rule* F^{SM} *is a SWF on domain* $\mathscr{S}$*.*

- If n is odd and preferences are single-peaked and strict, pairwise majority voting yields a transitive social preference relation.
- **Possibility result**:
  If n is odd, there exists a SWF on $\mathscr{S}$ that satisfies WP, IIA, and ND.
- If n is even, one can show that pairwise majority voting yields social preferences R whose strict component P is transitive. $\rightarrow$ *quasitransitive*

Proof.

Suppose n is odd and $(R_1, \dots, R_n) \in \mathscr{S}$. Let $R = F^{SM}(R_1, \dots, R_n)$. Clearly, R is complete. We have to show that R is also transitive.

- Since individual preferences are strict and n is odd, for all distinct $x, y \in X$, either $x \; P \; y$ or $y \; P \; x$.
- Suppose $x \; P \; y$ and $y \; P \; z$ for some $x, y, z \in X$. We will show that $x \; P \; z$.
  - Because preferences on X are single-peaked and strict, they are also single-peaked on $X' := \{x, y, z\} \subseteq X$.
  - $\implies$ there is a Condorcet winner in X'.
  - y and z cannot be Condorcet winners (because $x \; P \; y$ and $y \; P \; z$).
  - $\implies$ x is a Condorcet winner $\implies$ $x \; P \; z$. □

2.5 Quasi-transitive social welfare functions

- Another way to escape Arrow's impossibility result: relax the rationality requirements made on the social preference relation.
- A social preference relation R is **quasi-transitive** if its strict component is transitive, i.e., for all $x, y, z \in X$, $x \; P \; y$ and $y \; P \; z$ implies $x \; P \; z$.
  - Note: it is possible that $x \; I \; y$ and $y \; I \; z$ but $x \; P \; z$

Definition

A **quasi-transitive social welfare function (QSWF)** on domain $\mathscr{R}^n$ is a function $\tilde{F}$ that assigns a complete and quasi-transitive social preference relation $\tilde{F}(R_1, \ldots, R_n)$ to each profile of preferences $(R_1, \ldots, R_n) \in \mathscr{R}^n$.

Oligarchies

Definition

A QSWF $\tilde{F}$ is **oligarchic** if there is a group of agents (oligarchs) $O \subseteq N$ such that for all $x, y \in X$, $x \; R \; y$ whenever $x \; R_o \; y$ for at least one $o \in O$.

- $x \; P \; y$ if and only if $x \; P_o \; y$ for all oligarchs. Each oligarch has *veto power*.

Oligarchies with $|O| \geq 2$ are QSWFs that satisfy WP, IIA, and ND.

- If $|O| = 1$, there is a dictator. If $O = N$, the QSWF is anonymous.
- If there are few oligarchs, they resemble dictators. If there are many oligarchs, society is very indecisive.

Theorem (Gibbard, 1969)

Let $|X| \geq 3$. If a QSWF on domain $\mathscr{R}^n$ satisfies WP and IIA, then it is oligarchic.

For a proof see, e.g., Austen-Smith and Banks (2000, Sect. 2.3).

3 Social Choice Functions

3.1 Social choice functions and incentives

Two concerns regarding our approach to collective decisions using SWFs:

- Why do we need a full social preference relation when want to *choose* just one alternative?
- Individuals' preferences are usually **private information** that has to be elicited from them (as we have argued in Parts I and II).

In the following, we study aggregation of preferences into one **social choice** and focus on the problem of *eliciting* individual preferences.

Examples: elect a president or decide on public projects using a voting rule, choose a matching using some algorithm,...

Social choice functions

Definition

A **social choice function (SCF)** on domain $\mathscr{A} \subseteq \mathscr{R}^n$ is a mapping $f : \mathscr{A} \to X$ that assigns a chosen element $f(R_1, \ldots, R_n) \in X$ to every profile of individual preferences in $\mathscr{A}$.

- A SCF is a systematic method for transforming *individual preferences* into a *social decision*.

- We have already used this concept extensively in Part II, where we usually restricted the domain to quasi-linear preferences.
- Also the matching algorithms in Part I can be seen as SCFs.

Examples of SCFs

1. *Condorcet consistent rules*:

   A SCF is **Condorcet consistent** if it selects a Condorcet winner whenever a Condorcet winner exists.

2. *Scoring rules*:

   Consider strict preferences, $|X| = m$, and scores $s = (s_1, \ldots, s_m)$.
   A SCF f is a **scoring rule** if $f(R_1, \ldots, R_n) \; F^S(R_1, \ldots, R_n) \; y$ for all $y \in X$.

3. *Dictatorship*

Definition

A social choice function f is **dictatorial** if there is an agent i such that f always chooses one of i's most preferred alternatives.

Plurality voting and related SCFs

Plurality voting:

- Each agent votes for her favorite alternative. The alternative with the most votes is chosen.
- Special case of scoring rule: $s_1 = 1$ and $s_k = 0$ for all $k \geq 2$.

Plurality with runoff:

- 1st round: each agent votes for her favorite alternative
- 2nd round: use simple majority voting to choose between the two alternatives with the most votes in the 1st round

Plurality with elimination ("instant runoff"):

- each agent votes for her favorite alternative;
  eliminate the alternative with the fewest votes
- repeat until only one alternative is left

Properties of SCFs: unanimity, efficiency

Definition

A SCF f is **unanimous** if $f(R_1, \ldots, R_n) = x$ whenever $x \; P_i \; y$ for each agent i and each alternative $y \neq x$.

- In practice, it would be difficult to commit to a decision rule that may fail to choose an alternative that is clearly the best outcome for everyone.

Definition

A SCF f on domain $\mathscr{A}$ is Pareto **efficient** if for each profile $(R_1, \ldots, R_n) \in \mathscr{A}$, there is no alternative $x \in X$ such that $x \; R_i \; f(R_1, \ldots, R_n)$ for each agent i and $x \; P_i \; f(R_1, \ldots, R_n)$ for at least one agent i.

- Note: efficiency implies unanimity.

Example: Borda count

Suppose $N = \{1, 2, 3\}$, $X = \{w, x, y, z\}$, and the **Borda count**, i.e., the scoring rule with $s = (4, 3, 2, 1)$ is used as the SCF.

R_1	R_2	R_3
x	x	y
y	y	x
w	w	w
z	z	z

$\tilde{R}_3$
y
w
z
x

- The Borda count yields $f(R_1, R_2, R_3) = x$.
- Suppose agent 3 strategically reports preferences $\tilde{R}_3$ instead of her true preferences R_3. As $f(R_1, R_2, \tilde{R}_3) = y$, agent 3 is strictly better off.

Incentives

If individuals are **privately informed** about their preferences, the collective decision must be based on (directly or indirectly) *reported* preferences.

Depending on the SCF, there are incentives to misrepresent preferences!

Why is this a problem?

- Fairness: strategic sophistication may differ across individuals
- When a particular SCF is used because of some desirable properties (such as Pareto efficiency), these properties are only guaranteed to hold with respect to the *reported* rather than the true preferences.

We will focus on **strategy-proof** SCFs:

- dominant strategy incentive compatible
- prior free, robust

Strategy-proofness

Definition

Let $\mathscr{B} \subseteq \mathscr{R}$. A social choice function f on $\mathscr{B}^n$ is **strategy-proof** if for all agents $i \in N$ and all profiles $(R_1, \ldots, R_n) \in \mathscr{B}^n$,

$$f(R_i, R_{-i}) \; R_i \; f(\tilde{R}_i, R_{-i}) \quad \text{for all } \tilde{R}_i \in \mathscr{B}.$$

- In Part II we called strategy-proof SCFs *truthfully implementable in dominant strategies*.
- If for a profile $(R_1, \ldots, R_n)$ and an agent i, there is a ranking $\tilde{R}_i$ such that $f(\tilde{R}_i, R_{-i}) \; P_i \; f(R_i, R_{-i})$, then i *can manipulate* f *at* $(R_1, \ldots, R_n)$. A SCF is strategy-proof if no agent can ever manipulate it.
- Clearly, there are always dictatorial SCFs that are strategy-proof. What about other SCFs?

3.2 The Gibbard-Satterthwaite theorem

A SCF f on $\mathscr{A}$ is **surjective** (or onto) if for every $x \in X$, there exists a profile $(R_1, \ldots, R_n) \in \mathscr{A}$ such that $f(R_1, \ldots, R_n) = x$.

Theorem (Gibbard, 1973; Satterthwaite, 1975)

Suppose $|X| \geq 3$. If a SCF on domain $\mathscr{P}^n$ is surjective and strategy-proof, then it is dictatorial.

- The same result can also be shown if indifference is possible (domain $\mathscr{R}^n$).
- A corollary: If $|X| \geq 3$, any *Pareto efficient* (or *unanimous*) and strategy-proof SCF is dictatorial (as efficient $\Rightarrow$ unanimous $\Rightarrow$ surjective).
- If a strategy-proof SCF is not surjective but its range contains at least three alternatives, then it is dictatorial for those alternatives.

Proof

See also Jehle and Reny (2011, Subsect. 6.5) or Gaertner (2009, Subsect. 5.2).

We will prove the theorem by showing the following:

1. If f is strategy-proof, then f is *monotonic*.
2. If f is surjective and *monotonic*, then f is *unanimous*.
3. If f is *monotonic* and *unanimous*, then f is dictatorial.

Definition

A social choice function f on $\mathscr{P}^n$ is **monotonic** if whenever $f(R_1, \ldots, R_n) = x$ and $x\ \tilde{P}_i\ y$ if $x\ P_i\ y$ for all $y \in X$ and $i \in N$, then $f(\tilde{R}_1, \ldots, \tilde{R}_n) = x$.

Lemma

If a SCF f on $\mathscr{P}^n$ is strategy-proof, then f is monotonic.

Proof.

Let $(R_1, \ldots, R_n) \in \mathscr{P}^n$ and suppose $f(R_1, \ldots, R_n) = a$.
Consider $(\tilde{R}_1, \ldots, \tilde{R}_n)$ such that for all $i \in N$ and $x \in X$, $a \; \tilde{P}_i \; x$ if $a \; P_i \; x$.
We will show that $f(\tilde{R}_1, \ldots, \tilde{R}_n) = a$.

- *Strategy-proofness for agent 1 implies $f(\tilde{R}_1, R_{-1}) = a$.*

  By contradiction, suppose $f(\tilde{R}_1, R_{-1}) = b \neq a$.
  Strategy-proofness at (R_1, R_{-1}) implies $a \; P_1 \; b$ and hence $a \; \tilde{P}_1 \; b$.
  Strategy-proofness at $(\tilde{R}_1, R_{-1})$ implies $b \; \tilde{P}_1 \; a$, a contradiction.

- *Strategy-proofness for agent 2 implies $f(\tilde{R}_1, \tilde{R}_2, R_{-1,2}) = a$.*

  ...

- *Strategy-proofness for agent n implies $f(\tilde{R}_1, \ldots, \tilde{R}_n) = a$.* □

Lemma

If a SCF f on $\mathscr{P}^n$ is surjective and monotonic, then f is unanimous.

Proof.

- Because f is surjective, for any $x \in X$,
  there is a profile $(R_1, \ldots, R_n) \in \mathscr{P}^n$ such that $f(R_1, \ldots, R_n) = x$.
- Construct a new profile $(R'_1, \ldots, R'_n)$ by taking $(R_1, \ldots, R_n)$ and moving x to the top of each individual ranking.
  $f(R_1, \ldots, R_n) = x$ and monotonicity imply $f(R'_1, \ldots, R'_n) = x$.
- For any $(R''_1, \ldots, R''_n)$ where x is at the top of each individual ranking,
  $f(R'_1, \ldots, R'_n) = x$ and monotonicity imply $f(R''_1, \ldots, R''_n) = x$. □

Lemma

If a SCF f on $\mathscr{P}^n$ is monotonic and unanimous, then f is dictatorial.

The proof of this lemma proceeds in five steps.

The following fact, implied by monotonicity, will be used several times.

Fact A: Consider a profile $(R_1, \ldots, R_n) \in \mathscr{P}^n$ where $f(R_1, \ldots, R_n) = x$ and some agents $O \subseteq N$ rank x just above y. For each $i \in O$, let $\tilde{R}_i$ be identical to R_i except that now i ranks y just above x. Then $f(\tilde{R}_O, R_{-O}) \in \{x, y\}$.

- By contradiction, suppose that $f(\tilde{R}_O, R_{-O}) = z \notin \{x, y\}$. As the ranking of z with respect to any other alternative is the same under $(\tilde{R}_O, R_{-O})$ and under (R_O, R_{-O}), monotonicity implies $f(R_O, R_{-O}) = z$. $\Rightarrow$ contradiction.

Step 1: Consider any two alternatives $a, b \in X$. Let $\mathbf{R}^0 = (R_1, \ldots, R_n) \in \mathscr{P}^n$ such that a is ranked highest and b is ranked lowest by every agent.

- Unanimity implies $f(\mathbf{R}^0) = a$.
- Change agent 1's ranking by raising b in it one position at a time. By monotonicity, the social choice remains a as long as $a \; P_1 \; b$. By *Fact A*, once b moves from the second to the first position in 1's ranking, the social choice either remains a or changes to b.
- If the social choice has remained a, raise b in agent 2's ranking. Continue until for some agent k the social choice becomes b. By unanimity, such an agent k exists.
- $\mathbf{R}^1$ is the profile just before and $\mathbf{R}^2$ is the profile just after the change.

Profile $\mathbf{R}^1$:

R_1	$\dots$	R_{k-1}	R_k	R_{k+1}	$\dots$	R_n
b		b	a	a		a
a		a	b	.		.
.		.	.	.		.
.		.	.	.		.
.		.	.	b		b

$\Rightarrow f(\mathbf{R}^1) = a$

Profile $\mathbf{R}^2$:

R_1	$\dots$	R_{k-1}	R_k	R_{k+1}	$\dots$	R_n
b		b	b	a		a
a		a	a	.		.
.		.	.	.		.
.		.	.	.		.
.		.	.	b		b

$\Rightarrow f(\mathbf{R}^2) = b$

Profile $\mathbf{R}^3$:

R_1	$\dots$	R_{k-1}	R_k	R_{k+1}	$\dots$	R_n
b		b	a	.		.
.		.	b	.		.
.		.	.	.		.
.		.	.	a		a
a		a	.	b		b

$\Rightarrow f(\mathbf{R}^3) = a$

Profile $\mathbf{R}^4$:

R_1	$\dots$	R_{k-1}	R_k	R_{k+1}	$\dots$	R_n
b		b	b	.		.
.		.	a	.		.
.		.	.	.		.
.		.	.	a		a
a		a	.	b		b

$\Rightarrow f(\mathbf{R}^4) = b$

Step 2: Construct $\mathbf{R}^3$ from $\mathbf{R}^1$ and $\mathbf{R}^4$ from $\mathbf{R}^2$ by moving a down to the lowest position for $i < k$ and to the second lowest position for $i > k$.

We will show that $f(\mathbf{R}^3) = a$ and $f(\mathbf{R}^4) = b$.

- $f(\mathbf{R}^2) = b$ and monotonicity imply $f(\mathbf{R}^4) = b$.
- $f(\mathbf{R}^4) = b$ and *Fact A* imply that $f(\mathbf{R}^3) \in \{a, b\}$.
- Suppose $f(\mathbf{R}^3) = b$. Then monotonicity would imply $f(\mathbf{R}^1) = b$, resulting in a contradiction. Hence, $f(\mathbf{R}^3) = a$.

Step 3: Consider a third alternative $c \in X$ and the profile $\mathbf{R}^5$.

- In $\mathbf{R}^5$, the position of a relative to any other alternative is as in $\mathbf{R}^3$.
- $f(\mathbf{R}^3) = a$ and monotonicity imply $f(\mathbf{R}^5) = a$.

Profile $\mathbf{R}^5$:

R_1	$\dots$	R_{k-1}	R_k	R_{k+1}	$\dots$	R_n
.		.	a	.		.
.		.	c	.		.
.		.	b	.		.
.		.	.	.		.
c		c	.	c		c
b		b	.	a		a
a		a	.	b		b

$\Rightarrow f(\mathbf{R}^5) = a$

Profile $\mathbf{R}^6$:

R_1	$\ldots$	R_{k-1}	R_k	R_{k+1}	$\ldots$	R_n
.		.	a	.		.
.		.	c	.		.
.		.	b	.		.
.		.	.	.		.
c		c	.	c		c
b		b	.	b		b
a		a	.	a		a

$\Rightarrow f(\mathbf{R}^6) = a$

Step 4: Construct $\mathbf{R}^6$ from $\mathbf{R}^5$ by reversing the order of a and b for $i > k$.

- $f(\mathbf{R}^5) = a$ and *Fact A* imply $f(\mathbf{R}^6) \in \{a, b\}$.
- Suppose $f(\mathbf{R}^6) = b$. Note that every agent ranks c above b under $\mathbf{R}^6$. Hence, $f(\mathbf{R}^6) = b$ and monotonicity would imply that the choice remains b even if we move c to top of every agent's ranking.
  $\implies$ contradicts unanimity. $\implies$ $f(\mathbf{R}^6) = a$.

Step 5: Consider any profile $\mathbf{R}$ where a is at the top of agent k's ranking.

- $f(\mathbf{R}^6) = a$ and monotonicity imply that $f(\mathbf{R}) = a$.
- Agent k is a dictator for alternative a.
- As a was arbitrary in Step 1, for each $a \in X$, there is a dictator for a.
- There cannot be distinct dictators for distinct alternatives.
  $\implies$ There is a single dictator for all alternatives.

□

Discussion

The Gibbard-Satterthwaite theorem yields a general **impossibility result**:

There is no surjective SCF on $\mathscr{R}^n$ that is strategy-proof *and* non-dictatorial.

Impossibility need not hold in **restricted environments**:

- Single-peaked preferences
- For the matching environments of Part I,
  there are SCFs that are non-dictatorial and strategy-proof:
  - top trading cycle algorithm
  - deferred acceptance algorithm if the receiving side is non-strategic
- Quasi-linear preferences in Part II:
  - many examples of non-dictatorial and strategy-proof SCFs
  - non-dictatorial SCFs with truthful reports for weaker equilibrium notion (BNE)

Moreover, agents may not always possess sufficiently detailed information to identify profitable manipulations. $\rightarrow$ truthful preference revelation may be an *equilibrium* for some non-dictatorial SCFs that *perform reasonably well*.

3.3 Relation to Arrow's theorem

Two different impossibility results:

- Arrow's theorem for non-dictatorial SWFs
- Gibbard-Satterthwaite theorem for non-dictatorial SCFs

The two results are closely related!

- More precisely, for the case of **strict preferences** there is a one-to-one relationship between SWFs that satisfy Arrow's axioms and SCFs that satisfy the assumptions of the Gibbard-Satterthwaite theorem.
- For another illustration of the close relationship between the two theorems, see Reny (2001) who provides a unified approach to proving both results.

One-to-one relationship for strict preferences

- A **strict Arrovian aggregator** is a SWF $F : \mathscr{P}^n \to \mathscr{P}$ that is WP and IIA.
- The SCF c_F **induced** by a strict Arrovian aggregator F is the mapping $c_F : \mathscr{P}^n \to X$ that chooses for each $(R_1, \ldots, R_n) \in \mathscr{P}^n$ the most preferred alternative according to $F(R_1, \ldots, R_n)$.
- Satterthwaite (1975) shows that the following holds if $|X| \geq 3$:
  1. For any strict Arrovian aggregator F, c_F is surjective and strategy-proof.
  2. For any SCF f on $\mathscr{P}^n$ that is surjective and strategy-proof, there is a unique strict Arrovian aggregator F such that $f = c_F$.
- For strict preferences, Arrow's theorem and the Gibbard-Satterthwaite theorem are hence equivalent. In particular, Arrow's conditions ensure that the social choice process cannot be manipulated.

3.4 Single-peaked preferences

Consider the restricted domain of **single-peaked** preference profiles:

- There is a linear order $>$ on X.
- Each agent i's preferences R_i are single-peaked with respect to $>$.
- Let $\mathscr{S}_> \subset \mathscr{R}$ denote the set of all R_i that are single-peaked wrt $>$.

Median voter theorem:

- There always exists a Condorcet winner.
- If n is odd, the Condorcet winner is unique and equal to the **median peak**.

  The median peak is denoted by $\mathrm{med}(x_1^*, \ldots, x_n^*)$ and defined as the $m \in X$ that satisfies

$$\left|\{i \in N : m \geq x_i^*\}\right| \geq \tfrac{n+1}{2} \quad \text{and} \quad \left|\{i \in N : m \leq x_i^*\}\right| \geq \tfrac{n+1}{2}.$$

Condorcet consistent SCFs

Condorcet consistent SCFs on domain $\mathscr{S}^n_>$:

- If n is odd, there is a unique Condorcet consistent SCF:

$$f(R_1, \ldots, R_n) = \mathrm{med}(x_1^*, \ldots, x_n^*).$$

- If n is even, there are multiple Condorcet consistent SCFs, depending on which Condorcet winner is chosen when there are several.
  - Example: always select the smallest Condorcet winner.

$\rightarrow$ Special cases of *generalized median voting rules*.

Generalized median voting rules

- Idea: Combine the n reported peaks of real agents with $n-1$ exogenously fixed peaks of **phantom voters**.

  Define the SCF as the median of those $2n-1$ peaks.

Definition

A SCF f on $\mathscr{S}_>^n$ is a **generalized median voting rule (GMVR)** if there are $n-1$ *peaks of phantom voters* $y_1, \ldots, y_{n-1} \in X$ such that

$$f(R_1, \ldots, R_n) = \mathrm{med}(x_1^*, \ldots, x_n^*, y_1, \ldots, y_{n-1}).$$

for each profile $(R_1, \ldots, R_n) \in \mathscr{S}_>^n$.

- If all agents have the same peak, this peak is chosen ($\Rightarrow$*unanimity*).
- Otherwise, the phantom peaks act as arbitrators.

Examples of GMVRs

Let $X = \{a^1, \ldots, a^m\}$ such that $a^1 < a^2 < \cdots < a^m$.

1. **Condorcet consistent rules**: $y_l = a^1$ for $l < \frac{n}{2}$ and $y_l = a^m$ for $l > \frac{n}{2}$.

   If n is odd, $\mathrm{med}(x_1^*, \ldots, x_n^*, y_1, \ldots, y_{n-1}) = \mathrm{med}(x_1^*, \ldots, x_n^*)$.

   If n is even, which Condorcet winner is chosen depends on $y_{\frac{n}{2}}$.

   - If $y_{\frac{n}{2}} = a^1$ ($y_{\frac{n}{2}} = a^m$), SCF chooses smallest (greatest) Condorcet winner.

2. "**Positional dictators**": $y_l = a^1$ for $l < k$ and $y_l = a^m$ for $l \geq k$.

   SCF chooses kth highest peak.

   - if $k = 1$, maximum/"rightist" rule; if $k = n$, minimum/"leftist" rule

3. **Status quo rules**: $y_l = a^q$ for all l.

   can only move away from status quo a^q if no agent has peak there (vetoes)

GMVR: Pareto efficiency

Generalized median voting rules on domain $\mathscr{S}_{>}^{n}$ are Pareto **efficient**.

- Let $m = \mathrm{med}(x_1^*, \dots, x_n^*, y_1, \dots, y_{n-1})$.
- As there are n real peaks and only $n-1$ phantom peaks,

$$\begin{aligned} x_i^* &\leq m \quad \text{for at least one } i \in N \text{ and} \\ x_j^* &\geq m \quad \text{for at least one } j \in N. \end{aligned}$$

- $\Rightarrow$ For any alternative $x > m$, $m \; P_i \; x$ and for any $x < m$, $m \; P_j \; x$.
  $\Rightarrow$ no Pareto improvement possible

GMVR: (Group) strategy-proofness

Definition

A SCF f on $\mathscr{B}^n$ is **group strategy-proof** if for all profiles $(R_1, \ldots, R_n) \in \mathscr{B}^n$, there is no coalition $C \subseteq N$ and $\tilde{R}_C \in \mathscr{B}^{|C|}$ such that

$$f(\tilde{R}_C, R_{N\setminus C}) \; R_i \; f(R_C, R_{N\setminus C}) \quad \text{for all } i \in C$$
$$\text{and } f(\tilde{R}_C, R_{N\setminus C}) \; P_i \; f(R_C, R_{N\setminus C}) \quad \text{for some } i \in C.$$

Theorem

Generalized median voting rules on domain $\mathscr{S}^n_>$ are group strategy-proof.

- **Possibility result** for single-peaked preferences:
  GMVRs are strategy-proof and efficient SCFs that are not dictatorial.

Proof.

Consider a GMVR f and a profile $(R_1, \ldots, R_n) \in \mathscr{S}_>^n$ such that $f(R_1, \ldots, R_n) = a$. We will show that there is no coalition C that can manipulate the social choice to $b \neq a$ such that $b \; R_i \; a$ for all $i \in C$.

- Agents i with $x_i^* = a$ cannot be part of C (as $a \; P_i \; b$ for all $b \neq a$).
- Suppose there are $i, j \in C$ such that $x_i^* < a < x_j^*$. Then any shift to $b > a$ ($b < a$) would make agent i (agent j) worse off.
- Suppose $x_i^* < a$ for all $i \in C$. Then C can (potentially) only shift the social choice to some $b > a$, which would make all $i \in C$ worse off.
- Suppose $x_i^* > a$ for all $i \in C$. Then C can (potentially) only shift the social choice to some $b < a$, which would make all $i \in C$ worse off. □

Characterization of GMVRs

- Moulin (1980) shows that GMVRs are the only strategy-proof SCFs that are also *anonymous* and *Pareto efficient*.
- A SCF f is **anonymous** if $f(R_1, \ldots, R_n) = f(R_{\pi(1)}, \ldots, R_{\pi(n)})$ for all permutations π and all profiles $(R_1, \ldots, R_n)$.

Theorem (Moulin, 1980)

A SCF on domain $\mathscr{S}_>^n$ is strategy-proof, anonymous, and Pareto efficient ***if and only if*** *it is a generalized median voting rule.*

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