Revenue Maximization in Multi-Object Auctions

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April 25, 2017

Literature

- Myerson R. (1981): ”Optimal Auction Design”, *Mathematics of Operations Research* **6**, 58-73.
- Manelli, A and Vincent, D. (2007): "Multidimensional Mechanism Design: Revenue Maximization and the Multiple-Good Monopoly", *Journal of Economic Theory* **137**, 153-185.

Multi-Object Auction Model

- n agents and m objects. Value of object j to agent i is $v_i^j \in R_+$.
- Value of a bundle of objects is the sum of the values of the objects in the bundle.
- The utility of agent i who obtains set of objects S and makes monetary transfer t is

$$\sum_{j \in S} v_i^j - t.$$

- Auctioneer maximize revenues = maximize the sum of monetary transfers made by agents.

Information

- Each agent i knows $\mathbf{v}_i = (v_i^1, .., v_i^m)$ but regards $\mathbf{v}_k$, $k \neq i$, as being a random draw from a compact and convex set $V_k \subset R_+^m$.
- $\mathbf{v}_k$ independent of $\mathbf{v}_l,\ l \neq k$.
- Distribution F_k with density $f_k > 0$ is common knowledge
- Auctioneer regards all values as random variables.

Mechanisms

- In a *direct revelation mechanism* (DRM) each agent i reports $\mathbf{v}_i$ (not necessarily truthfully!), and the mechanism determines a physical and monetary allocation.
- Formally, a mechanism consists of:

1. $\forall i, j,\ p_i^j(\mathbf{v}_1, .., \mathbf{v}_n) \in [0, 1]$ is the probability that agent i gets object j when the reports are $(\mathbf{v}_1, ..\mathbf{v}_n)$. Feasibility requires that

$$\forall j, \mathbf{v}_1, .., \mathbf{v}_n,\ \sum_{i=1}^{n} p_i^j(\mathbf{v}_1, .., \mathbf{v}_n) \leq 1$$

2. $\forall i,\ t_i(\mathbf{v}_1, .., \mathbf{v}_n) \in R$ is the monetary transfer made by agent i to the auctioneer when reports are $(\mathbf{v}_1, ..\mathbf{v}_n)$.

The Revelation Principle

- In a Bayes-Nash equilibrium, each agent optimizes given his expectation about the actions of others and expectations are correct.
- The Bayes-Nash equilibrium of **any** possible selling scheme can be replicated by a DRM where it is a Bayes-Nash equilibrium for each agent to truthfully reveal his values.
- Thus, in order to find the revenue maximizing selling scheme it is enough to restrict attention to truthful (or incentive compatible) DRM's

Incentive Compatibility

- Fix a DRM $(\mathbf{p}, \mathbf{t})$ and define:

$$\begin{aligned} q_i^j(\widehat{\mathbf{v}}_i) &= \int_{V_{-i}} p_i^j(\mathbf{v}_{-i}, \widehat{\mathbf{v}}_i) f_{-i}(\mathbf{v}_{-i}) d\mathbf{v}_{-i} \\ T_i(\widehat{\mathbf{v}}_i) &= \int_{V_{-i}} t_i(\mathbf{v}_{-i}, \widehat{\mathbf{v}}_i) f_{-i}(\mathbf{v}_{-i}) d\mathbf{v}_{-i} \end{aligned}$$

  Agent $i's$ expected utility is given by :

$$U_i(\mathbf{v}_i, \widehat{\mathbf{v}}_i) = \mathbf{q}_i(\widehat{\mathbf{v}}_i) \cdot \mathbf{v}_i - T_i(\widehat{\mathbf{v}}_i)$$

- Truthful telling requires then that for all i,and for all $\mathbf{v}_i, \widehat{\mathbf{v}}_i \in V_i$ we have:

$$S_i(\mathbf{v}_i) := U_i(\mathbf{v}_i, \mathbf{v}_i) = \sup_{\widehat{\mathbf{v}}_i \in V_i} U_i(\mathbf{v}_i, \widehat{\mathbf{v}}_i)$$

Consequences of IC

- The function $S_i(\mathbf{v}_i) = U_i(\mathbf{v}_i, \mathbf{v}_i)$ is the supremum of linear functions, hence it is continuous and convex, and $\nabla S_i = \mathbf{q}_i$ a.e.
- $\mathbf{q}_i$ is cyclically monotone and

$$S_i(\mathbf{v}_i) = S_i(\underline{\mathbf{v}}_i) + \int_{\underline{\mathbf{v}}_i}^{\mathbf{v}_i} \mathbf{q}_i(\mathbf{w}_i) \cdot d\mathbf{w}_i$$

  where $\underline{\mathbf{v}}_i$ is an arbitrary vector in V_i, and where the integral does not depend on the path of integration.
- We can write:

$$\begin{aligned} T_i(v_i) &= -S_i(\mathbf{v}_i) + \mathbf{q}_i(\mathbf{v}_i) \cdot \mathbf{v}_i \\ &= T_i(\underline{\mathbf{v}}_i) - q_i(\underline{\mathbf{v}}_i) \cdot \underline{\mathbf{v}}_i + \mathbf{q}_i(\mathbf{v}_i) \cdot \mathbf{v}_i - \int_{\underline{\mathbf{v}}_i}^{\mathbf{v}_i} \mathbf{q}_i(\mathbf{w}_i) \cdot d\mathbf{w}_i \end{aligned}$$

Payoff Equivalence

- In other words, in a truthful DRM, the expected monetary transfer T_i is completely determined, up to a constant, by the expected probabilities of getting the various objects $\mathbf{q}_i$ (and hence by the primitive probabilities p_i^j).
- **This important result is called the payoff-equivalence theorem**.
- Conversely, any mechanism $(\mathbf{p}, \mathbf{t})$ that satisfies the above conditions is a truthful DRM.

Revenue Maximization: Individual Rationality

- Since $S_i(\mathbf{v}_i) = S_i(\underline{\mathbf{v}}_i) + \int_{\underline{v}_i}^{v_i} \mathbf{q}_i(\mathbf{w}_i) \cdot d\mathbf{w}_i$ and $\mathbf{q}_i$ is non-negative we obtain that
$$S_i(\underline{\mathbf{v}}_i) \geq 0 \Rightarrow S_i(\mathbf{v}_i) \geq 0, \forall \mathbf{v}_i.$$
- Let assume now for simplicity that $V_i = [0,1]^m$, and let us take $\underline{\mathbf{v}}_i = (0,0,..0)$. Then, $S_i(\underline{\mathbf{v}}_i) = -T_i(\underline{\mathbf{v}}_i)$, so that
$$S_i(\underline{\mathbf{v}}_i) \geq 0 \Leftrightarrow T_i(\underline{\mathbf{v}}_i) \leq 0.$$
- In other words, the agent with zero values for all objects cannot be asked to make a positive transfer to the auctioneer.
- Thus, in a revenue maximizing mechanism we have $T_i(\underline{\mathbf{v}}_i) = 0$.

Expected Revenue

We have:

$$\begin{aligned} T_i(\mathbf{v}_i) &= \nabla S_i(\mathbf{v}_i) \cdot \mathbf{v}_i - S_i(\mathbf{v}_i) \\ &= \mathbf{q}_i(\mathbf{v}_i) \cdot \mathbf{v}_i - \int_{\underline{v}_i}^{\mathbf{v}_i} \mathbf{q}_i(\mathbf{w}_i) \cdot d\mathbf{w}_i \end{aligned}$$

The expected revenue is then given by

$$\begin{aligned} R &= \sum_{i=1}^{n} \left[\int_{V_i} T_i(\mathbf{v}_i) f_i(\mathbf{v}_i) d\mathbf{v}_i \right] \\ &= \sum_{i=1}^{n} \left[\int_{V_i} \left[\mathbf{q}_i(\mathbf{v}_i) \cdot \mathbf{v}_i - \int_{\underline{\mathbf{v}}_i}^{\mathbf{v}_i} \mathbf{q}_i(\mathbf{w}_i) \cdot d\mathbf{w}_i \right] f_i(\mathbf{v}_i) d\mathbf{v}_i \right] \end{aligned}$$

Constraints

- The above revenue can be written directly in terms of the mechanism's primitive probabilities p_i^j.
- Thus the auctioneer maximizes R over all functions $p_i^j(\mathbf{v}_1, ..\mathbf{v}_n)$ such that:

1. $\forall i, j, \mathbf{v}_1, .., \mathbf{v}_n, \quad p_i^j(\mathbf{v}_1, ..\mathbf{v}_n) \in [0, 1].$
2. $\forall j, \mathbf{v}_1, .., \mathbf{v}_n, \quad \sum_{i=1}^{n} p_i^j(\mathbf{v}_1, .., \mathbf{v}_n) \leq 1.$
3. $\forall i$, the vector field $\mathbf{q}_i$ defined by

$$q_i^j(\mathbf{v}_i) = \int_{V_{-i}} p_i^j(\mathbf{v}_{-i}, \mathbf{v}_i) f_{-i}(\mathbf{v}_{-i}) d\mathbf{v}_{-i}$$

is the gradient of a convex function.

Remarks

- Myerson (1981) completely solves the 1-object case.
- Heuristic: ignore constraint 3; solve pointwise; find conditions on the distribution under which constraint 3 is satisfied. Solution is a deterministic allocation.
- Manelli and Vincent (2007) have shown that the extreme points in the multidim. case need not be deterministic