Literature

Multi-Object Auction Model

- $n$ agents and $m$ objects. Value of object $j$ to agent $i$ is $v_j^i \in R_+$. 
- Value of a bundle of objects is the sum of the values of the objects in the bundle.
- The utility of agent $i$ who obtains set of objects $S$ and makes monetary transfer $t$ is
  \[ \sum_{j \in S} v_j^i - t. \]
- Auctioneer maximize revenues = maximize the sum of monetary transfers made by agents.
Information

- Each agent $i$ knows $v_i = (v^1_i, \ldots, v^m_i)$ but regards $v_k$, $k \neq i$, as being a random draw from a compact and convex set $V_k \subset R^m$.
- $v_k$ independent of $v_l$, $l \neq k$.
- Distribution $F_k$ with density $f_k > 0$ is common knowledge.
- Auctioneer regards all values as random variables.
In a *direct revelation mechanism* (DRM) each agent $i$ reports $v_i$ (not necessarily truthfully!), and the mechanism determines a physical and monetary allocation.

Formally, a mechanism consists of:

1. $\forall i, j, \ p_i^j(v_1, \ldots, v_n) \in [0, 1]$ is the probability that agent $i$ gets object $j$ when the reports are $(v_1, \ldots, v_n)$. Feasibility requires that
   \[
   \forall j, v_1, \ldots, v_n, \sum_{i=1}^{n} p_i^j(v_1, \ldots, v_n) \leq 1
   \]

2. $\forall i, \ t_i(v_1, \ldots, v_n) \in R$ is the monetary transfer made by agent $i$ to the auctioneer when reports are $(v_1, \ldots, v_n)$.
The Revelation Principle

- In a Bayes-Nash equilibrium, each agent optimizes given his expectation about the actions of others and expectations are correct.
- The Bayes-Nash equilibrium of any possible selling scheme can be replicated by a DRM where it is a Bayes-Nash equilibrium for each agent to truthfully reveal his values.
- Thus, in order to find the revenue maximizing selling scheme it is enough to restrict attention to truthful (or incentive compatible) DRM’s.
Incentive Compatibility

- Fix a DRM \((p, t)\) and define:

\[
q_i^j(\hat{v}_i) = \int_{V_{-i}} p_i^j(v_{-i}, \hat{v}_i)f_{-i}(v_{-i})dv_{-i}
\]

\[
T_i(\hat{v}_i) = \int_{V_{-i}} t_i(v_{-i}, \hat{v}_i)f_{-i}(v_{-i})dv_{-i}
\]

Agent \(i's\) expected utility is given by :

\[
U_i(v_i, \hat{v}_i) = q_i(\hat{v}_i) \cdot v_i - T_i(\hat{v}_i)
\]

- Truthful telling requires then that for all \(i\), and for all \(v_i, \hat{v}_i \in V_i\) we have:

\[
S_i(v_i) := U_i(v_i, v_i) = \sup_{\hat{v}_i \in V_i} U_i(v_i, \hat{v}_i)
\]
Consequences of IC

- The function $S_i(v_i) = U_i(v_i, v_i)$ is the supremum of linear functions, hence it is continuous and convex, and $\nabla S_i = q_i$ a.e.
- $q_i$ is cyclically monotone and

$$S_i(v_i) = S_i(v_i) + \int_{v_i}^{v_i} q_i(w_i) \cdot d w_i$$

where $v_i$ is an arbitrary vector in $V_i$, and where the integral does not depend on the path of integration.

- We can write:

$$T_i(v_i) = -S_i(v_i) + q_i(v_i) \cdot v_i$$

$$= T_i(v_i) - q_i(v_i) \cdot v_i + q_i(v_i) \cdot v_i - \int_{v_i}^{v_i} q_i(w_i) \cdot d w_i$$
In other words, in a truthful DRM, the expected monetary transfer $T_i$ is completely determined, up to a constant, by the expected probabilities of getting the various objects $q_i$ (and hence by the primitive probabilities $p_i$).

This important result is called the payoff-equivalence theorem.

Conversely, any mechanism $(p, t)$ that satisfies the above conditions is a truthful DRM.
Since $S_i(v_i) = S_i(v_i) + \int_{v_i}^{V_i} q_i(w_i) \cdot dw_i$ and $q_i$ is non-negative we obtain that

$$S_i(v_i) \geq 0 \Rightarrow S_i(v_i) \geq 0, \forall v_i.$$ 

Let assume now for simplicity that $V_i = [0, 1]^m$, and let us take $v_i = (0, 0, ..0)$. Then, $S_i(v_i) = -T_i(v_i)$, so that

$$S_i(v_i) \geq 0 \iff T_i(v_i) \leq 0.$$ 

In other words, the agent with zero values for all objects cannot be asked to make a positive transfer to the auctioneer.

Thus, in a revenue maximizing mechanism we have $T_i(v_i) = 0$. 

Expected Revenue

We have:

\[
T_i(v_i) = \nabla S_i(v_i) \cdot v_i - S_i(v_i) = q_i(v_i) \cdot v_i - \int_{v_i}^{v_i} q_i(w_i) \cdot dw_i
\]

The expected revenue is then given by

\[
R = \sum_{i=1}^{n} \left[ \int_{V_i} T_i(v_i)f_i(v_i)dv_i \right] = \sum_{i=1}^{n} \left[ \int_{V_i} \left[ q_i(v_i) \cdot v_i - \int_{v_i}^{v_i} q_i(w_i) \cdot dw_i \right] f_i(v_i)dv_i \right]
\]
The above revenue can be written directly in terms of the mechanism’s primitive probabilities $p^j_i$.

Thus the auctioneer maximizes $R$ over all functions $p^j_i(v_1, \ldots v_n)$ such that:

1. $\forall i, j, v_1, \ldots, v_n, \quad p^j_i(v_1, \ldots v_n) \in [0, 1]$.
2. $\forall j, v_1, \ldots, v_n, \quad \sum_{i=1}^{n} p^j_i(v_1, \ldots, v_n) \leq 1$.
3. $\forall i$, the vector field $q_i$ defined by

$$q^j_i(v_i) = \int_{V_{-i}} p^j_i(v_{-i}, v_i) f_{-i}(v_{-i}) dv_{-i}$$

is the gradient of a convex function.
Remarks

- Myerson (1981) completely solves the 1-object case.
- Heuristic: ignore constraint 3; solve pointwise; find conditions on the distribution under which constraint 3 is satisfied. Solution is a deterministic allocation.
- Manelli and Vincent (2007) have shown that the extreme points in the multidim. case need not be deterministic.