Quasi-Linear Utility

- \( x = (k, t_1, .., t_I) \), where: \( k \in K \) (physical outcomes, "projects"), 
  \( t_i \in \mathbb{R} \) (money)
- \( u_i(x, \theta_i) = v_i(k, \theta_i) + t_i \)
- \( f(\theta) = f(\theta_1, ..\theta_I) = (k(\theta), t_1(\theta), .., t_I(\theta)) \)

**Definition**

Efficient SCF \( f^*(\theta) = (k^*(\theta), t_1^*(\theta), .., t_I^*(\theta)) \):

1. Value maximization: \( \forall \theta, \ k^*(\theta) \in \arg \max_k \sum_i v_i(k, \theta_i) \)
2. Budget Balance: \( \forall \theta, \ \sum_i t_i^*(\theta) = 0 \)
Example: Allocation of indivisible good

- Indivisible good owned by seller
- \( I \) buyers
- \( k = (y_1, .., y_I) \) where \( y_i \in \{0, 1\} \) and \( \sum y_i \leq 1 \)
- \( v_i(k, \theta_i) = v_i((y_1, .., y_I), \theta_i) = y_i \theta_i \)
- Efficient allocation: \( y_i = 1 \) if \( \theta_i \in \arg \max_j \theta_j \); all monetary transfers from buyers go to seller
The Vickrey-Clarke-Groves (VCG) Mechanism

- Direct Revelation Mechanism
- $k(\theta) = k^*(\theta)$ (value maximization)
- $t_i^*(\theta) = \sum_{j \neq i} v_j(k^*(\theta), \theta_j) + h_i(\theta_{-i})$, where $h_i$ is arbitrary

**Theorem**

The VCG mechanism truthfully implements the value maximizing SCF in dominant strategies
The Pivot Mechanism

Problem

VCG mechanism requires huge transfers to the agents.

Solution

appropriate definition of the $h_i$ functions

- Denote by $k_{-i}^*(\theta_{-i})$ the value maximizing project in the absence of $i$
- Define

  $$t_i^*(\theta) = \sum_{j \neq i} v_j(k^*(\theta), \theta_j) + h_i(\theta_{-i})$$

  $$= \sum_{j \neq i} v_j(k^*(\theta), \theta_j) - \sum_{j \neq i} v_j(k_{-i}^*(\theta_{-i}), \theta_j)$$

- Exercise: Prove that: $\forall \theta, \sum_i t_i^*(\theta) \leq 0$
Example: Allocation of Indivisible Object

- Efficient allocation: \( y_i = 1 \) if \( \theta_i \in \arg \max_j \theta_j \)
- In VCG mechanism:
  \[
t_i^*(\theta) = \begin{cases} 
0 + h_i(\theta_{-i}), & \text{if } i = \arg \max \theta_j \\
\arg \max \theta_j + h_i(\theta_{-i}), & \text{otherwise}
\end{cases}
\]
- In pivot mechanism:
  \[
t_i^*(\theta) = \begin{cases} 
- \arg \max_{j \neq i} \theta_j, & \text{if } i = \arg \max \theta_j \\
0, & \text{otherwise}
\end{cases}
\]
- Second price auction!
Agent 1 is seller, owns indivisible object, value for object $\theta_1$

Agent 2 is buyer, value for object $\theta_2$

Values are distributed independently on interval $[0, 1]$ according to densities $\phi_1, \phi_2$.

VCG Mechanism:

$$k^*(\theta) = k^*(\theta_1, \theta_2) = \begin{cases} 1, & \text{if } \theta_1 \geq \theta_2 \\ 2, & \text{otherwise} \end{cases}$$

$$t_1^*(\theta) = \begin{cases} 0 + h_1(\theta_2), & \text{if } \theta_1 \geq \theta_2 \\ \theta_2 + h_1(\theta_2), & \text{otherwise} \end{cases}$$

$$t_2^*(\theta) = \begin{cases} \theta_1 + h_2(\theta_1), & \text{if } \theta_1 \geq \theta_2 \\ 0 + h_2(\theta_1), & \text{otherwise} \end{cases}$$
Budget Balance:

\[ t_1^*(\theta) + t_2^*(\theta) = 0 \Rightarrow \]
\[
\int_0^1 \int_0^1 [t_1^*(\theta) + t_2^*(\theta)] \phi_1(\theta_1) \phi_2(\theta_2) d\theta_1 d\theta_2 = 0 \Rightarrow
\]
\[
H_1 + H_2 + \int_0^1 \int_0^1 \max[\theta_1, \theta_2] \phi_1(\theta_1) \phi_2(\theta_2) d\theta_1 d\theta_2 = 0
\]

where \( H_i = E_{\theta_i} h_i \). Noting that \( \theta_1 < \max[\theta_1, \theta_2] \) a.e., this yields:

\[
H_1 + H_2 < -E\theta_1
\]

With positive probability
Participation Constraints:

Highest Seller Type: \[ 1 + H_1 \geq 1 \Rightarrow H_1 \geq 0 \]

Lowest Buyer Type: \[ E\theta_1 + H_2 \geq 0 \Rightarrow H_2 \geq -E\theta_1 \]

This yields:
\[ H_1 + H_2 \geq -E\theta_1 \]

a contradiction!
Bayesian Implementation and Payoff Equivalence

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Summer Term 2015
First-Price Auction

- n bidders; i.i.d types
- \( F(\theta_i), \varphi(\theta_i) > 0 \)

- Symmetric Equilibrium \( b_i(\theta_i) = b(\theta_i) \) increasing and differentiable.
First-Price Auction

\[ U_i(\theta_i, \hat{\theta}_i) = [\theta_i - b(\hat{\theta}_i)] \cdot F(\hat{\theta}_i)^{n-1} \]

\text{FOC: } -b'(\hat{\theta}_i)F(\hat{\theta}_i)^{n-1} + (n - 1)(\theta_i - b(\hat{\theta}_i))F(\hat{\theta}_i)^{n-2}\varphi(\hat{\theta}_i) = 0

\[ b'(\theta_i) = (n - 1)(\theta_i - b(\theta_i))\frac{\varphi(\theta_i)}{F(\theta_i)} \]

\[ b(\theta_i) = \theta_i \]
First-Price Auction

Example

- \( n = 2, \theta_i \sim U[0, 1] \)
- \( b'(\theta_i) = \frac{1}{\theta_i} (\theta_i - b(\theta_i)); \ b(0) = 0 \)
- Solution: \( b(\theta_i) = \frac{1}{2} \theta_i \)
- Expected Utility: \( (\theta_i - \frac{1}{2} \theta_i) \cdot \theta_i = \frac{1}{2} \theta_i^2 \)
- Second Price: \( (\theta_i - E[\theta_j|\theta_j \leq \theta_i]) \theta_i = \frac{1}{2} \theta_i^2 \)
- Revenue: \( \frac{1}{3} \)
Myerson's Auction Model

- 1 object, n bidders
- independent, private values
- $\theta_i$ distributed: $F_i(\theta_i), \varphi_i(\theta_i) > 0$
- Revelation Mechanism $p_i(\theta) \in [0, 1], \forall i; \sum_{i=1}^{n} p_i(\theta) \leq 1$
- $t_i(\theta) \in \mathbb{R}, \forall i$
Myerson’s Auction Model

- \( q_i(\theta_i) = E_{\theta_{-i}} p_i(\theta_i, \theta_{-i}) \)
- \( T_i(\theta_i) = E_{\theta_{-i}} t_i(\theta_i, \theta_{-i}) \)
- \( U_i(\theta_i, \hat{\theta}_i) = \theta_i q_i(\hat{\theta}_i) + T_i(\hat{\theta}_i) \)

**Bidder i’s problem:**

\[ \max_{\hat{\theta}_i} U_i(\theta_i, \hat{\theta}_i) = U_i(\theta_i, \theta_i) \]

_Truth-telling condition!_

- Denote \( \bar{U}_i(\theta_i) = U_i(\theta_i, \theta_i) \)
Myerson’s Auction Model

Theorem (Myerson)

A mechanism \( \{p_i, t_i\}_{i=1}^n \), is Bayesian incentive compatible if and only if:

1. \( q_i \) is increasing in \( \theta_i \)

2. \( \bar{U}_i(\theta_i) = \bar{U}_i(\underline{\theta}_i) + \int_{\underline{\theta}_i}^{\theta_i} q_i(s) ds \), where \( \bar{U}_i(\theta_i) = U_i(\theta_i, \theta_i) \)
Proof.

1) $\hat{\theta}_i > \theta_i$

\[
\hat{\theta}_i q_i(\hat{\theta}_i) + T_i(\hat{\theta}_i) \geq \hat{\theta}_i q(\theta_i) + T_i(\theta_i)
\]

\[
\theta_i q_i(\theta_i) + T_i(\theta_i) \geq \theta_i q_i(\hat{\theta}_i) + T_i(\hat{\theta}_i)
\]

\[
(\hat{\theta}_i - \theta_i) q_i(\hat{\theta}_i) \geq (\hat{\theta}_i - \theta_i) q_i(\theta_i)
\]

$\bar{U}_i(\theta_i)$ is maximum of affine functions

$\rightarrow$ convex, equal integral of its derivative.
Proof. (cont.)

2) ← assume w.l.o.g. \( \theta_i > \hat{\theta}_i \)

\[
\tilde{U}_i(\theta_i) - U_i(\theta_i, \hat{\theta}_i) \\
= \tilde{U}_i(\theta_i) - \tilde{U}_i(\hat{\theta}_i) - q(\hat{\theta}_i)[\theta_i - \hat{\theta}_i] \\
= \int_{\hat{\theta}_i}^{\theta_i} q(s)ds - q(\hat{\theta}_i)[\theta_i - \hat{\theta}_i] \\
= \int_{\hat{\theta}_i}^{\theta_i} [q(s) - q(\hat{\theta}_i)]ds \geq 0
\]
Consequence:

\[ \tilde{U}_i(\theta_i) = \bar{U}_i(\theta_i) + \int_{\theta_i}^{\theta} q(s) ds \]

\[ = \theta_i q_i(\theta_i) + T_i(\theta_i) \]

\[ \Rightarrow T_i(\theta_i) = -\theta_i q_i(\theta_i) + \int_{\theta_i}^{\theta} q_i(s) ds + \bar{U}_i(\theta_i) \]

Payoff and revenue equivalence!
The Optimal Auction

\[
\max \{ p_i, t_i \}_{i=1}^n E_\theta \left( \sum_{i=1}^n T_i(\theta_i) \right) \]

1. \( p_i(\theta) \in [0, 1]; \sum_{i=1}^n p_i(\theta) \leq 1, \forall \theta \)

2. \( q_i(\theta_i) \) is monotone

Observation: \( \bar{U}_i(\theta_i) = 0 \) is optimal
The Optimal Auction

\[
- \int_{\theta_i}^{\bar{\theta}_i} T_i(\theta_i) \varphi_i(\theta_i) d\theta_i
\]

\[= \int_{\theta_i}^{\bar{\theta}_i} \theta_i q_i(\theta_i) - \int_{\theta_j}^{\theta_i} q_i(s) ds \varphi_i(\theta_i) d\theta_i \]

\[= E_\theta[\theta_i p_i(\theta)] - \int_{\theta_i}^{\bar{\theta}_i} \int_{\theta_i}^{\theta_i} q_i(s) ds \varphi_i(\theta_i) d\theta_i \]
\[
\int_{\bar{\theta}^i}^{\tilde{\theta}^i} \left[ \int_{\theta^i}^{\theta^i} q_i(s)ds \right] \varphi_i(\theta_i) d\theta_i
\]

\[
= \int_{\theta^i}^{\tilde{\theta}^i} q_i(s)ds - \int_{\theta^i}^{\tilde{\theta}^i} q_i(\theta_i)F_i(\theta_i) d\theta_i
\]

\[
= \int_{\theta^i}^{\tilde{\theta}^i} q_i(\theta_i)\left[ \frac{1-F_i(\theta_i)}{\varphi_i(\theta_i)} \right] \varphi_i(\theta_i) d\theta_i
\]
The Optimal Auction

\[ = E_\theta(p_i(\theta)\left[\frac{1-F_i(\theta_i)}{\phi(\theta_i)}\right]) \]

to conclude:

\[ -E_\theta(T_i(\theta_i)) = E_\theta[p_i(\theta_i)(\theta_i - \frac{1-F_i(\theta_i)}{\phi_i(\theta_i)})] \]

\[ E_\theta(\sum_{i=1}^n - T_i(\theta_i)) = E_\theta[\sum_{i=1}^n p_i(\theta)(\theta_i - \frac{1-F_i(\theta_i)}{\phi(\theta_i)})] \]
Assumption: \( J_i(\theta_i) = \theta_i - \frac{1 - F_i(\theta_i)}{\varphi_i(\theta_i)} \) increasing

Revenue Maximization:

\[
p_i(\theta) = \begin{cases} 
1, & \{ i \in \arg \max_j J_j(\theta_j) \} \land \{ J_i(\theta_i) \geq 0 \} \\
0, & \text{otherwise}
\end{cases}
\]

Satisfies monotonicity constraint!
Assumption: \( F_i = F, \forall i \)

Result: second-price auction with reservation price that satisfies

\[
R^* - \frac{1 - F(R^*)}{\varphi(R^*)} = 0
\]

is optimal!
Equivalence between Bayesian and Dominant Strategy Incentive Compatible Mechanisms

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Summer Term 2015
A Problem in Discrete Tomography

Problem

When does a 0 – 1 matrix with given row and column sums exist?

- Consider row sum (3, 2, 2, 1, 1) and two different column sums:

<table>
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<tr>
<th>1</th>
<th>1</th>
<th>0</th>
<th>0</th>
<th>0</th>
<th>1</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>2</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>2</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>4</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

  | 1 | 2! | 0 | 0 | 0 | 0 | 3 |
  | 1 | 1 | 0 | 0 | 0 | 2 |
  | 1 | 1 | 0 | 0 | 0 | 2 |
  | 1 | 0 | 0 | 0 | 0 | 1 |
  | 1 | 0 | 0 | 0 | 0 | 1 |
  | 5 | 4 | 0 | 0 | 0 |

- Matrix exist if the vector of column sums is "less diverse" than the vector (5, 3, 1, 0, 0).
- See Gale (1957), and Ryser (1957) for the general result. Variations (continuous case, densities) are in Kellerer (1961) and Strassen (1965).
FIGURE 1- In this sequence of images (axial (a), coronal (b), sagittal (c) and 3D-CT (d)), it may be observed an anterior inferior orbit wall fracture (arrows). An arrow is simultaneously pointed by the software in all the images. This fracture cannot be detect in axial image.
The Monotone Lift

Problem

When unique reconstruction is not possible, are there solutions with special properties?

Theorem (Gutmann et al. (1991))

Let \( \phi = \phi(x_1, x_2, \ldots x_n) \) be measurable on \([0, 1]^n\) with \(0 \leq \phi \leq 1\). Assume that the one-dimensional marginals

\[
\Phi_i(x_i) = \int \phi(x_1, x_2, \ldots x_n) \, dx_{-i}
\]

are non-decreasing in \(x_i\), \(i = 1, 2, \ldots n\). Then there exists \(\psi\) measurable on \([0, 1]^n\) such that \(0 \leq \psi \leq 1\), \(\psi\) has the same marginals as \(\phi\), and moreover, \(\psi\) is non-decreasing in each coordinate.
Monotone Lift: Example

Example

\[
\phi = \begin{bmatrix} 2 & 4 & 4 & 10 \\ 4 & 2 & 6 & 12 \\ 4 & 6 & 4 & 14 \\ 10 & 12 & 14 \end{bmatrix} \quad \implies \quad \psi = \begin{bmatrix} 2 & 4 & 4 & 10 \\ 4 & 4 & 4 & 12 \\ 4 & 4 & 6 & 14 \\ 10 & 12 & 14 \end{bmatrix}
\]

Note that \( \sum_{i,j} (\psi_{ij})^2 \leq \sum_{i,j} (\phi_{ij})^2 \).
• $K$ social alternatives and $N$ agents. The utility of agent $i$ in alternative $k$ is given by $a_i^k x_i + c_i^k + t_i$ where $x_i \in [0, 1]$ is agent $i$’s private type, where $a_i^k, c_i^k \in \mathbb{R}$ with $a_i^k \geq 0$, and where $t_i \in \mathbb{R}$ is a monetary transfer.

• Types are drawn independently of each other, according to strictly increasing distributions $F_i$. Type $x_i$ is private information of agent $i$.

• Manelli and Vincent assume: $K = N$ ; $a_i^j = 1$, $a_i^j = 0$ for any $j \neq i$; $c_i^k = 0$ for any $i, k$. 
A direct revelation mechanism (DRM) $M$ is given by $K$ functions $q^k : [0, 1]^N \rightarrow [0, 1]$ and $N$ functions $t_i : [0, 1]^N \rightarrow \mathbb{R}$ where $q^k(x_1, ..., x_N)$ is the probability with which alternative $k$ is chosen, and $t_i(x_1, ..., x_N)$ is the transfer to agent $i$ if the agents report types $x_1, ..., x_N$.

A DRM $M$ is *Dominant-Strategy Incentive Compatible* (DIC) if truth-telling constitutes a dominant strategy equilibrium in the game defined by $M$ and the given utility functions. A DRM $M$ is *Bayes-Nash Incentive Compatible* (BIC) if truth-telling constitutes a Bayes-Nash equilibrium in the game defined by $M$ and the given utility functions.
Incentive Compatible Mechanisms II

Fact

A necessary condition for $M$ to be DIC is that, for each agent $i$, and for any signals of others, the function $\sum_{k=1}^{K} a^k_i q^k(x_1, \ldots, x_N)$ is non-decreasing in $x_i$. Moreover, any $K$ functions $q^k$ that satisfy this condition are part of a DIC mechanism.

Fact

A necessary condition for $M$ to be BIC is that, for each agent $i$, the function $\sum_{k=1}^{K} a^k_i Q^k_i(x_i)$ is non-decreasing, where

$$\forall i, k, \quad Q^k_i(\hat{x}_i) = \int_{[0,1]^{N-1}} q^k(x_1, \ldots, x_i, \hat{x}_i, x_{i+1}, \ldots, x_N) dF_{-i},$$

is the expected probability that alternative $k$ is chosen if agents $j \neq i$ report truthfully while agent $i$ reports type $\hat{x}_i$. Moreover any $K$ functions $q^k$ that satisfy this condition are part of a BIC mechanism.
Equivalent Mechanisms

**Definition**

1. Two mechanisms \( M \) and \( \tilde{M} \) are \( P \)-equivalent if, for each \( i, k \) and \( x_i \), it holds that \( Q^k_i(x_i) = \tilde{Q}^k_i(x_i) \), where \( Q^k_i \) and \( \tilde{Q}^k_i \) are the conditional expected probabilities associated with \( M \) and \( \tilde{M} \), respectively.

2. Two mechanisms \( M \) and \( \tilde{M} \) are \( U \)-equivalent if they provide the same interim utilities for each agent \( i \) and each type \( x_i \) of agent \( i \).

- For each agent \( i \), interim utility is obtained (up to a constant) by integrating the function \( \sum_{k=1}^{K} a^k_i Q^k_i(x_i) \) with respect to \( x_i \) - this is the **Payoff Equivalence Theorem**. Thus \( P \)-equivalence implies \( U \)-equivalence.
Since $q^2(x_1, \ldots, x_N) = 1 - q_1^1(x_1, \ldots, x_N)$, we have

$$
\sum_{k=1}^{2} a_i^k Q_i^k(x_i) = a_i^2 + (a_i^1 - a_i^2) Q_i^1(x_i),
$$

and therefore $U$-equivalence implies $P$-equivalence (the two notions coincide).

**Theorem**

Assume that $K = 2$. Then for any BIC mechanism there exists a $P$-equivalent (and thus $U$-equivalent) DIC mechanism.
Theorem

Assume that $a_i^k = a_j^k = a^k$ for all $k, i, j$, and that $F_i = F$ for all $i$. Moreover, assume that $0 = a^1 \leq a^2 \leq \cdots \leq a^K = 1$. Then for any symmetric, BIC mechanism there exists an U-equivalent symmetric DIC mechanism.

Proof shows how to achieve U-equivalence using only the 2 alternatives with highest and lowest slope, respectively. Thus, U-equivalence does not necessarily ensure that the ex-ante probabilities of different alternatives are preserved.
$v_i(k, \theta_i) = \theta_i^k$

Types - ind. distributed

**Vector Field:** $q_i^k(\theta_i) = E_{\theta-i}[p_i^k(\theta)]$

$T_i(\theta_i) = E_{\theta-i}[t_i(\theta)]$

$U_i(\theta_i, \hat{\theta}_i) = \theta_i \ast q_i(\hat{\theta}_i) + T_i(\hat{\theta}_i)$
Theorem (Jehiel, Moldovanu, Stacchetti, *JET* (1999))

The mechanism \( \{ \{ p_i^k \}_{k \in K}, t_i \}_{i \in I} \) is Bayes-Nash incentive compatible if and only if:

1. \[ q_i(\theta_i) - q_i(\hat{\theta}_i) \cdot [\theta_i - \hat{\theta}_i] \geq 0 \]
2. \[ \hat{U}_i(\theta_i) = \hat{U}_i(\underline{\theta}_i) + \int_{\underline{\theta}_i}^{\theta_i} q_i(s) \cdot ds, \forall i, \theta_i \]

*Path independence!*
Example

- two objects
- 1 buyer, valuations $\theta_A, \theta_B$
- Mechanism: $P_A \cdot P_B, P_{AB} < P_A + P_B$
Observation

\[ k(\theta_i, \theta_{-i}) = k(\hat{\theta}_i, \theta_{-i}) \]
\[ \Rightarrow t_i(\theta_i, \theta_{-i}) = t_i(\hat{\theta}_i, \theta_{-i}) \]

Lemma

\[ k(\theta_i, \theta_{-i}) = l, \quad k(\hat{\theta}_i, \theta_{-i}) = \hat{l} \]
\[ \Rightarrow \hat{\theta}_i^l - \hat{\theta}_i^l \geq \hat{\theta}_i^l - \theta_i^l \quad \forall \theta_i, \hat{\theta}_i, \theta_{-i} \]

Weak monotonicity!
Proof.

\[ \theta_i^l + t_i(\theta_i, \theta_{-i}) \geq \hat{\theta}_i^l + t_i(\hat{\theta}_i, \theta_{-i}) \]

\[ \hat{\theta}_i^l + t_i(\hat{\theta}_i, \theta_{-i}) \geq \hat{\theta}_i^l + t_i(\theta_i, \theta_{-i}) \]

⇒ \[ \theta_i^l + \hat{\theta}_i^l \geq \hat{\theta}_i^l + \hat{\theta}_i^l \]

⇒ \[ \hat{\theta}_i^l - \theta_i^l \geq \hat{\theta}_i^l - \theta_i^l \]
Theorem (Saks-Yu)

Assume types spaces are convex, and consider \( k(\theta) \). There exist \( t_1(\theta), \ldots, t_I(\theta) \) such that \( f(\theta) = [k(\theta), t_1(\theta), \ldots, t_I(\theta)] \) is truthfully implementable in dominant strategies if and only if \( k(\theta) \) is weakly monotone.
Affine Maximizers

\[ \alpha_1, \ldots, \alpha_l \in \mathbb{R}_+ \]
\[ R_1, \ldots, R_K \in \mathbb{R} \]

\[ k(\theta) \in \arg \max_k \left[ \sum_{i=1}^{l} \alpha_i V_i(k, \theta_i) + R_k \right] \]
Theorem (Roberts)

Let $\Theta_i = \mathbb{R}^{|K|}$ and $|K| \geq 3$. Then $f(\theta)$ is truthfully implementable in dominant strategies if and only if the associated $k(\theta)$ is an affine maximizer.
Interdependent Values

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Summer Term 2015
Sellers with cars of quality \( \theta \in [\underline{\theta}, \bar{\theta}] \)

\[ F(\theta); F'(\theta) = f(\theta) > 0 \]

Utility = \[
\begin{cases} 
  \theta - P, & \text{buyer} \\
  R(\theta) + P, & \text{seller} 
\end{cases}
\]
Equilibrium Condition

\[ P(\theta) = \theta \]

\[ S(P) = \{ \theta | R(\theta) \leq \theta \} = D(P) \]

*Efficient Trade!*
Incomplete Information

\[ S(P) = \{ \theta | R(\theta) \leq P \} \]

\[ E(P) \equiv \mathbb{E}[\theta | \theta \in S(P)] \]

\[ D(P) = \begin{cases} 
0, & E(P) < P \\
[\theta, \bar{\theta}], & E(P) = P \\
[\underline{\theta}, \bar{\theta}], & E(P) > P 
\end{cases} \]

**Equilibrium**: \( P^* = E(P^*) \)
Example

$$R(\theta) = \frac{2}{3} \theta; \ F(\theta) = \theta \text{ on } [0, 1]$$

$$E[\theta|\frac{2}{3} \theta \leq P] = E[\theta|\theta \leq \frac{3}{2} P] = \frac{3}{4} P$$

$$P^* = \frac{3}{4} P^* \Rightarrow P^* = 0$$

No car is sold at all!
\[ u_i(x, \theta_1, \ldots, \theta_J) = V_i(k, \theta_1, \ldots, \theta_J) + t_i \]

VCG Transfer:
\[ t_i^*(\theta) = \sum_{j \neq i} V_j(k, \theta) + h_i(\theta_{-i}) \]

Problem: \( t_i^* \) depends on \( \theta_i \)!

Solution?
Example (1 Object, 2 Bidders)

\[ k = 1, 2 \]

\[ V_i(k, \theta) = \begin{cases} 0, & k \neq i \\ a\theta_i + b\theta_{-i}, & k = i \end{cases} \]

*where* \( a > b > 0 \)

\[ k^*(\theta) = \begin{cases} 1, & \theta_1 \geq \theta_2 \\ 2, & \theta_1 < \theta_2 \end{cases} \]
Example (1 Object, 2 Bidders)

\[\theta_1 = \theta_2\]

\[V_1(\theta_1)\]

\[V_2(\theta_1)\]

\[\ell^*(\theta) = -V_2(\hat{\theta}_1, \hat{\theta}_2)\]

\[= -(a + b) \theta_2\]
Important Condition

\[ \frac{\partial V_i(\theta)}{\partial \theta_i} > \frac{\partial V_j(\theta)}{\partial \theta_i}, \quad \forall i, j \]

Equilibrium Notion

Nash Equilibrium (ex-post)
Bilateral Bargaining

Seller: $V_S(\theta_S, \theta_B) \uparrow \theta_S$

Buyer: $V_B(\theta_S, \theta_B) \uparrow \theta_B$

$$\frac{\partial V_S}{\partial \theta_S} > \frac{\partial V_B}{\partial \theta_S}; \quad \frac{\partial V_B}{\partial \theta_B} > \frac{\partial V_S}{\partial \theta_B}$$
Bilateral Bargaining

**Definition**

\[ V_B(\theta_S, \theta_B^*(\theta_S)) = V_S(\theta_S, \theta_B^*(\theta_S)) \]
\[ V_S(\theta_S^*(\theta_B), \theta_B) = V_B(\theta_S^*(\theta_B), \theta_B) \]

\[ k^*(\theta) = \begin{cases} 
S, & V_S(\theta) \geq V_B(\theta) \\
B, & V_B(\theta) > V_S(\theta) 
\end{cases} \]
Bilateral Bargaining

Definition (cont.)

\[ t^*_B(\theta) = \begin{cases} 
0, & k^* = B \\
V_S(\theta_S, \theta^*_B(\theta_S)), & k^* = S 
\end{cases} \]

\[ t^*_S(\theta) = \begin{cases} 
0, & k^* = S \\
V_B(\theta^*_S(\theta_S), \theta_B), & k^* = B 
\end{cases} \]

*Modified VCG payments*
Conditions for efficient trade (Fieseler, Kittsteiner, Moldovanu, *JET* (2003))

\[
E_{\theta_S}[V_B(\theta_S, \theta_B)] \geq P \geq E_{\theta_B}[V_S(\bar{\theta}_S, \theta_B)]
\]

Akerlof’s case:

\[
E_{\theta_S}[V_B(\theta_S)] \geq P \geq V_S(\bar{\theta}_S)
\]

\[
\frac{1}{2} < \frac{2}{3}
\]
Multidimensional Types + Interdependent Values

\[ V_i(k, \theta_1, \cdots, \theta_J) \]

Example

- 2 agents, \( i = 1, 2 \)
- 2 alternatives, \( k = A, B \)
- \( V_i^k(\theta_1^k), i = 1, 2; k = A, B \)
- (Only agent 1 is informed)
Example (cont.)

Value Maximization

\[ k(\theta_1) \in \arg\max_k \sum_{i=1}^{2} V_i(k, \theta^k_1) \]

Incentives for agent 1:

\[ k(\theta_1) \in \arg\max_k [V_1(A, \theta^A_1) + t^A_1, V_1(B, \theta^B_1) + t^B_1] \]
Example (cont.)

\[ \sum v_i^A = \sum v_i^B \]
Example (cont.)

Congruence Condition

\[
\frac{\partial V_1(A, \theta_1^A)}{\partial \theta_1^A} \frac{\partial V_1(B, \theta_1^B)}{\partial \theta_1^B} = \frac{\partial}{\partial \theta_1^A} \left[ \sum_{i=1}^{2} V_i(A, \theta_1^A) \right] \frac{\partial}{\partial \theta_1^B} \left[ \sum_{i=1}^{2} V_i(B, \theta_1^B) \right]
\]

Non-generic condition!
Theorem (Jehiel et al., *Econometrica* (2001))

For generic utility functions only constant social choice functions are truthfully implementable in ex-post equilibrium.

Robust implementation is impossible!
Incentive compatibility ⇔ the vector field $q_i$ is

1) monotone
2) conservative

\[ \frac{\partial^2 q_i^k(\theta_i)}{\partial \theta_i^k \partial \theta_i^{k'}} = \frac{\partial^2 q_i^{k'}(\theta_i)}{\partial \theta_i^{k'} \partial \theta_i^k} \]
Mechanism Design Without Money

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Summer Term 2015
Usual setting with general utility function $U_i(x, \theta_i)$ - private values
let $R_i = \{\prec_i \mid \prec_i = \prec_i(\theta_i) \text{ for some } \theta_i\}$
ordinal preferences
**Definition**

\[ L_i(x, \prec_i) = \{y | y \prec_i x\} \]

A social choice function \( f \) is monotonic if

\[
\forall \theta, L_i(f(\theta), \theta_i) \subseteq L_i(f(\theta), \theta'_i) \forall i \implies f(\theta) = f(\theta') \quad \text{if} \quad (\theta_{-i}, \theta'_i)
\]
Theorem (Gibbard-Satterthwaite)

Assume that $X$ is finite with $|X| \geq 3$, and that $\forall i, P = R_i$, where $P$ is the set of all rational preferences (without indifference). Let $f$ be onto and dominant strategy incentive compatible; then $f$ is dictatorial!
Proof GS (Sketch).

1. $R_i = P$ and $f$ DIC $\Rightarrow f$ is monotonic
2. $R_i = P$ and $f$ DIC and onto $\Rightarrow f$ is pareto efficient
3. $f$ is monotonic and efficient

$\Rightarrow f$ is dictatorial
Possibility results:

1. $|X| = 2$ - simple majority rule
2. single peaked preference
Theorem

Let preferences be single-peaked and let the number of agents be odd. Then the SCF that chooses at each profile the median peak is dominant strategy incentive compatible.
Theorem (Moulin 1980)

Consider a setting with $n$ agents, and let $R_i = SP \ \forall i$ (with respect to a given order $K$).
Let $f$ be DIC, anonymous and unanimous. Then there exist $n - 1$ phantom voters such that $f$ chooses the median of all voters’ peaks.
Voting on single alternatives in the SP order (say $1, 2, \cdots, K$). $k$ is chosen if the number of yes votes is at least $\tau(k)$, where $\tau(k) \geq \tau(k')$, $k \leq k'$. If no alternative passes threshold, $K$ is chosen.
Successive Voting With Thresholds

Sincere Strategy for voter with peak on $k$:

\[
\begin{pmatrix}
\text{No, No, \cdots, No} & \text{Yes, Yes, \cdots, Yes} \\
1, 2, \cdots, k - 1 & k, k + 1, \cdots, K
\end{pmatrix}
\]

Monotone strategy:

\[
\begin{pmatrix}
\text{No, No, \cdots, No} & \text{Yes, Yes, \cdots, Yes} \\
1, 2, \cdots, l & l + 1, \cdots, K
\end{pmatrix}
\]

Sincere $\Rightarrow$ monotone

Markov strategies
Successive Voting With Thresholds

Theorem

Assume that all players besides $i$ use monotone strategies. Then it is optimal for $i$ to use the sincere strategy. In particular, sincere voting is an ex-post perfect Nash equilibrium.
Successive Voting With Thresholds

Theorem

*Equivalence for single-peaked preferences.*

- Anonymous Unanimous \(\iff\) Successive voting with DIC mechanisms decreasing threshold \(\tau(k)\)
How many anonymous, unanimous, DIC mechanisms are there?

\[
\frac{(n+K-2)!}{(n-1)! (K-1)!}
\]
- 2 alternatives, A, B
- n mechanisms (supermajority)
- $l_A$ - number of phantoms on A
- $l_B = n - 1 - l_A$
- utility: $u^A(x), u^B(x), x \in [0, 1]$
- types $x$ i.i.d., distribution $F$
- single crossing:
Optimization

**Definition**

\[ u^A_L = E \left[ u^A(x) \mid x \leq x^{AB} \right] \]

\[ u^A_H = E \left[ u^A(x) \mid x \geq x^{AB} \right] \]

\[ u^B_L = E \left[ u^B(x) \mid x \leq x^{AB} \right] \]

\[ u^B_H = E \left[ u^B(x) \mid x \geq x^{AB} \right] \]
Optimization

\[ \gamma = \frac{u_L^A - u_L^B}{u_L^A - u_L^B + u_H^B - u_H^A} > 0 \]

First order conditions:
changing \( l_A \) to \( l_A + 1 \) or \( l_A - 1 \) should not be beneficial \( l_A = \lfloor n\gamma \rfloor \)
Two-Sided Matching
Gale-Shapley 1961

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Summer Term 2015
Two-Sided Matching

- \( M = \{ m_1, \cdots, m_n \} \)
- \( W = \{ w_1, \cdots, w_p \} \)
- \( M \cap W = \emptyset \)
- \( P(m_i) = w_j, w_k, \cdots m_i w_l \cdots \)
- \( P(w_j) = m_i, m_l, \cdots w_j m_n \cdots \)

Ranked ordinal lists
Two-Sided Matching

Matching:

\[ \mu : M \cup W \rightarrow M \cup W \]

1. \( \mu^2(x) = x, \ \forall x \in M \cup W \)
2. \( \mu(m) \in W \cup \{m\} \)
3. \( \mu(w) \in M \cup \{w\} \)

Individual rationality: \( \mu(x) \succeq^x x, \ \forall x \in M \cup W \)
Two-Sided Matching

**Stability:** $\mu$ is stable if:

1. $\mu$ is individually rational, and
2. $\notin \{m, w\}$ such that $\mu(m) \neq w$ and
   
   $w \succeq_m \mu(m)$
   
   $m \succeq_w \mu(w)$
Theorem (Gale, Shapley (1962))

*For any two-sided market, the set of stable matchings is non-empty.*

Proof.

Deferred acceptance algorithm *(Gale-Shapley).*
Two-Sided Matching

Example

\[ P(m_1) = w_2 w_1 w_3 \]
\[ P(m_2) = w_1 w_3 w_2 \]
\[ P(m_3) = w_1 w_2 w_3 \]

\[ P(w_1) = m_1 m_3 m_2 \]
\[ P(w_2) = m_3 m_1 m_2 \]
\[ P(w_3) = m_1 m_3 m_2 \]

\[ \mu^1 = \begin{array}{ccc} w_1 & w_2 & w_3 \\ m_1 & m_2 & m_3 \end{array} \]

\[ \mu^2 = \begin{array}{ccc} w_2 & w_1 & w_3 \\ m_1 & m_2 & m_3 \end{array} \]

Stable!
The Roommate Problem

\[ P(a) = bcd \]
\[ P(b) = cad \]
\[ P(c) = abd \]
\[ P(d) = abc \]

\[ \mu^1 = \begin{align*} ab \\ \mid \mid \\ cd \end{align*} \]

\[ \mu^2 = \begin{align*} a & \rightarrow b \\ c & \rightarrow d \end{align*} \]

\[ \mu^3 = \begin{align*} ab \\ \mid \mid \\ dc \end{align*} \]

None is stable!
Lattice Operator

\[ \mu \lor_M \mu'(m) = \begin{cases} 
\mu(m), & \text{if } \mu(m) \succeq_M \mu'(m) \\
\mu'(m), & \text{otherwise}
\end{cases} \]

\[ \mu \lor_M \mu'(w) = \begin{cases} 
\mu(w), & \text{if } \mu(w) \preceq_W \mu'(w) \\
\mu'(w), & \text{otherwise}
\end{cases} \]

and analogously for \( \land_M \).
Lemma (Conway)

Let $\mu, \mu'$ be stable matchings. Then $\mu \lor_M \mu'$ and $\mu \land_M \mu'$ are also stable matchings.

Consequence: M-Optimal and W-Optimal stable matchings.
Strategic Questions

Theorem

There is no mechanism such that:
1. It is a dominant strategy for all agents to state preferences truthfully.
2. A stable matching is chosen for every report.
Proof.

\[ M = \{ m_1 m_2 \} \]
\[ W = \{ w_1 w_2 \} \]
\[ P(m_1) = w_1 w_2 \]
\[ P(m_2) = w_2 w_1 \]
\[ \mu^M = \begin{pmatrix} m_1 & m_2 \\ w_1 & m_2 \end{pmatrix} \]
\[ P(w_1) = m_2 m_1 \]
\[ P(w_2) = m_1 m_2 \]
\[ \mu^W = \begin{pmatrix} m_1 & m_2 \\ w_2 & w_1 \end{pmatrix} \]

Suppose \( \Gamma(p) = \mu^M \)
Consider strategy \( Q(w_2) = m_1 \) then \( \Gamma(p_{-w_2}, Q(w_2)) = \mu^W \)
Strategic Questions

Theorem

Let $\Gamma = \Gamma^M$ be the mechanism that assigns to each profile of reports the $M$-optimal stable matching. Then truthful reporting is a dominant strategy for all men.
Consider a Nash equilibrium of $\Gamma^M$ where men use their dominant strategies (and women respond optimally). Then the outcome is a stable matching for the true preferences!
Proof.

\( P \) - True Preferences.

\( P' \rightarrow \) stated preferences.

Let \( \mu' = \Gamma(P') \). Assume \( \{m,w\} \) block \( \mu' \). Then, in \( \Gamma^M(p') \) \( m \) must have proposed to \( w \), and she rejected him. Consider \( P'(w) = m \Rightarrow \) then \( \{m,w\} \) would be matched. Contradiction to \( P' \) being an equilibrium.
Many-to-one Matching

- **n** firms: \( F_i = \{ F_1, \cdots, F_n \} \)
- **m** workers: \( W = \{ W_1, \cdots, W_m \} \)
- matching: \( \mu : F \cup W \rightarrow 2^{F \cup W} \)

1. \( |\mu(w)| = 1, \forall w \in W \)
2. \( \mu(w) = w, \text{ if } \mu(w) \notin F \)
3. \( \mu(F) \subseteq W \ [\mu(F) = \emptyset] \)
4. \( \mu(w) = F_i \iff w \in \mu(F_i) \)
Many-to-one Matching

- \( P(w_i) = F_i, F_j, \cdots, F_k, w_i, \cdots \)
- \( P(F_j) = S_1.S_2.\cdots, \emptyset, \cdots \)
- where \( S_i \subseteq W \) (sets of workers)

Definition

\( Ch_F(S) = S' \iff \forall S'' \subseteq S, \text{ it holds that } S' \prefr F S''. \)
Many-to-one Matching

Matching is blocked by \( \{w, F\} \) if:

1. \( \mu(w) \neq F \)
2. \( F \succ_w \mu(w) \)
3. \( w \in Ch_F(\mu(F) \cup \{w\}) \)

Individual rationality:

\[
\begin{align*}
\mu(w) & \succeq_w w \\
\mu(F) = Ch_F(\mu(F)) & \succ_F \emptyset
\end{align*}
\]
Many-to-one Matching

Definition

Firms have substitutable preferences if: \( \forall S, w, w' \in S \)
\( w \in Ch_F(S) \Rightarrow w \in Ch_F(S \setminus w') \).

Theorem

If all firms have substitutable preferences, then the set of stable matchings is not empty.

Proof.

Run G-S algorithm with firms proposing.
The Assignment Game

- $M \cap W = \emptyset$
- $V(S) = 0$ if $S \subseteq M$, or $S \subseteq W$
- $V(S) = \max \sum_{m \in S} v(m, \mu^S(m))$
- where $\mu^S$ is a matching on coalition $S$. 
The Assignment Game

\[ \mu^* \text{ is optimal if } V(M \cup W) = \sum_{m \in M} v(m, \mu^*(m)) \]

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<table>
<thead>
<tr>
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<tr>
<td>(m_1)</td>
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<td>11</td>
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<tr>
<td>(m_1)</td>
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<td>8</td>
</tr>
<tr>
<td>(m_3)</td>
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Optimal matching generically unique.
The Assignment Game

Interpretation:

M-sellers, each has indivisible object; reserve price: $c_m$.

W-buyers, each wants at most one object; value for object m: $r_{wm}$.

$v(m, w) = \max\{0, r_{wm} - c_m\}$
Stable payoff vectors

1. feasible: $\sum_{i \in M \cup W} x_i \leq V(M \cup W)$
2. ind. rational: $x_i \geq 0, \forall i \in M \cup W$
3. no blocking: $x_m + x_w \geq v(m, w), \forall m, w$
### Example

<table>
<thead>
<tr>
<th></th>
<th>$w_1$</th>
<th>$w_2$</th>
</tr>
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<tbody>
<tr>
<td>$m_1$</td>
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<td>3</td>
</tr>
<tr>
<td>$m_2$</td>
<td>7</td>
<td>8</td>
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</tbody>
</table>

$\left(x_{m_1}, x_{w_1}, x_{m_2}, x_{w_2}\right)$

- $(2.5, 2.5, 4, 4)$
- $(1, 4, 3, 5)$

$\Downarrow$ *Stable*
The Assignment Game

Theorem (Shapley-Shubik)

*The set of stable payoff vectors is not empty.*

Proof.

\[
\min \left( \sum_{m \in M} x_m + \sum_{w \in W} x_w \right)
\]

s.t.

1. \( x_i \geq 0 \quad \forall i \in M \cup W \)
2. \( x_m + x_w \geq v(m, w), \forall m, w \)
Proof. (cont.)

Program has solution. Let $R$ be the minimum. Need to show feasibility $R \leq V(M \cup W)$.

Dual program

$$\max \sum_{i,j} p_{ij} v(i, j) \text{ s.t.}$$

1. $\sum_i p_{ij} \leq 1, \forall j$
2. $\sum_j p_{ij} \leq 1, \forall i$
3. $p_{ij} \geq 0, \forall i, j$
Proof. (cont.)

Dual has solution with $p_{ij} \in \{0, 1\}$, $p_{ij} = 1 \Rightarrow i \in M, j \in W$ or vice-versa. The max achieved equals $R$ (duality theorem). Obvious now that $R \leq V(M \cup W)$. 
Lemma

Let $x, y$ be stable payoff vectors. Define:

\[ u_m = \max(x_m, y_m), \forall m \in M \]

\[ u_w = \min(x_w, y_w), \forall w \in W \]

Then $u$ is also a stable payoff vector (and analogously for women).

Proof.

Exercise!
Consequence:
The set of stable payoff vectors is a lattice with a maximal and minimal element.

Proof.
Above Lemma + compactness
### Example (homogeneous objects)

The Assignment Game

<table>
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<table>
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<td>m₂</td>
<td>3</td>
<td>4</td>
</tr>
<tr>
<td>m₃</td>
<td>0</td>
<td>1</td>
</tr>
</tbody>
</table>

Competitive prices:

\[ 19 \leq P \leq 20 \]
The Assignment Game

Example

\[ (m_1, m_2, w_1, w_2) \quad P_1, P_2 \]

\[ (1, 0, 2, 3) \rightarrow (1, 0) \quad m_1 \]

\[ (3, 2, 0, 1) \rightarrow (3, 2) \quad m_2 \]

\[ w_1 \quad w_2 \]

\[ \begin{array}{cc}
3 & 4 \\
1 & 3 \\
\end{array} \]
The Assignment Game

Example (cont.)

Vickrey Prices
\[ P_1 = 1; P_2 = 0 \]

(each buyer pays externality)

\[ \begin{array}{ccc}
  & w_1 & w_2 \\
 m_1 & 3 & 4 \\
 m_2 & 1 & 3 \\
\end{array} \]

⇒ Outcome equivalent to minimal stable payoff vector (buyer optimal).
The Simultaneous Ascending Clock Auction

\( M \) - objects; \( W \) - buyers

\( c_m \) \( r_{wm} \)

(\( c_m \) and \( r_{wm} \) are integers)

Assume \( M \) contains dummy object \( m_0 \):

\( c_{m_0} = 0 \)

\( r_{wm_0} = 0, \forall w \)

Dummy can be assigned to several buyers
Let $P$ be a vector of prices, one for each object in $M$

$$D_w(P) = \{ m \in M | r_{wm} - P_m \geq \max_{m' \in M} (r_{wm'} - P_{m'}) \}$$

→ Demand of $w$ at $P$
Definition

P is quasi-competitive if there exists a matching \( \mu \) such that:

1. \( \mu(w) = m \Rightarrow m \in D_w(P) \)
2. \( \mu(w) = w \Rightarrow m_0 \in D_w(P) \)

Competitive equilibrium \((P, \mu)\):

1. quasi-competitive +
2. \( P_m = c_m \) if \( m \notin \mu(W) \)
Lemma

1. Every competitive equilibrium yields a stable payoff vector.
2. Every stable payoff vector can be obtained via a competitive equilibrium.

Proof.
Exercise.
Hall’s Theorem

Let $B, C$ be disjoint sets $\forall b \in B, D_b \subseteq C$

Can we find a matching $\mu : B \rightarrow C$ such that

1. $\forall b, \mu(b) \in D_b$

2. $\mu(b) \neq \mu(b'), \forall b, b'$
Hall’s Theorem

Necessary condition:

∀ B′ ⊆ B,

| \bigcup_{b \in B'} D_b | ≥ | B' |

Theorem

The above condition is also sufficient!
\[ P_m(i) = c_m, \forall m \]

Bidders announce \( D_w(P(1)) \)

Find \( \mu \) such that
\[ \mu(w) \in D_w(P(1)) \]

If possible → stop otherwise

\[ \exists W' \text{ such that} \]
\[ |W'| > \bigcup_{w \in W} D_w(P(1)) \]

⇒ There exists an overdemanded set of objects

Choose \( M' \subseteq M \) to be a minimal over-demanded set.
\[ P_m(2) = \begin{cases} 
P_m(1) + 1 & , m \in M' \\
P_m(1) & , otherwise 
\end{cases} \]

Continue...

Algorithm must stop since at high prices only \( m_0 \in \mathcal{D}_w(P(K)), \forall \mathcal{w} \)
The ascending auction stops at the minimum quasi-competitive price vector. In the corresponding direct revelation mechanism, truthfully revealing valuations is a dominant strategy for each buyer.