Mechanism Design and Social Choice
Part II: Mechanism Design Theory

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1 Introduction

1.1 Overview

Part I (Matching):

- incentives: one particular application of mechanism design
- settings without monetary transfers

Part II (Mechanism Design Theory):

- more explicit model of **private information** of individuals
- first introduce general concepts applicable to various settings
- then focus on settings with transferable utility (money)

Before introducing the general setting, we take a first look at some commonly used selling mechanisms: posted prices and auctions.

Literature

Main textbook reference:

- Mas-Colell, A., Whinston, M.D., and Green, J.R. (1995): *Microeconomic Theory*, Oxford University Press. → Chapter 23

Other useful textbooks:

- Börgers, T. (2015): *An Introduction to the Theory of Mechanism Design*, Oxford University Press.
- Krishna, V. (2010): *Auction Theory*, Academic Press.
- Milgrom, P.R. (2004): *Putting Auction Theory to Work*, Cambridge University Press.
- Jehle, G.A. and Reny, P.J. (2011): *Advanced Microeconomic Theory*, Financial Times Prentice Hall. → Chapter 9

1.2 Example: A Seller's Problem

- one seller, one buyer
- Seller owns a single indivisible object, valuation 0.
- Buyer has valuation v for object.
- Buyer privately knows v.
  Seller knows that v is drawn from distribution F with support $[0, 1]$.

Simple mechanism: **posted price**

- Seller sets price p, buyer decides whether or not to buy at that price.
- Seller's optimal posted price:

$$p^* \in \arg\max_p \,(1 - F(p))p$$

- Revenue maximization ($p = p^* > 0$) vs. efficient allocation ($p = 0$)

Could the Seller do better?

- Seller could use arbitrarily complicated selling procedure, e.g.,
  - negotiation (and renegotiation)
  - offer buyer lotteries at different prices

- Given value v, buyer optimally chooses actions in the selling procedure.

  $\rightarrow$ Any selling procedure results in the buyer obtaining the object with some probability $q(v)$ and paying some amount $t(v)$.

Revelation principle (informal): For every selling procedure, there is an incentive compatible direct selling mechanism that results in the same outcome.

Direct selling mechanism (q, t):

1. Buyer reports valuation $\tilde{v}$.
2. Buyer obtains object with probability $q(\tilde{v})$ and pays $t(\tilde{v})$.

(q, t) is **incentive compatible** if it is optimal for the buyer to report the true v.

Optimal direct selling mechanism

Revelation principle greatly simplifies seller's problem:
Can restrict search for optimal selling procedure to direct selling mechanisms!

At this stage, we will not solve the full problem (we will come back to it later).
$\rightarrow$ Restrict attention to $q(v) \in \{0, 1\}$, i.e., *deterministic allocation*.

Incentive compatibility (IC):

$$q(v)v - t(v) \geq q(v')v - t(v') \quad \text{for all } v, v'.$$

Implications of IC:

1. $q(v)$ is non-decreasing.
   - If $q(v) = 1$ is incentive compatible for v, then we must have $q(v') = 1$ for all $v' > v$.
2. $t(v) = t(v')$ for all v, v' where $q(v) = q(v')$.
   - If $t(v) > t(v')$, v would gain from imitating v'.

$\Rightarrow$ Incentive compatible direct selling mechanisms take the following from:
For some $\hat{v} \in [0,1]$ and some t_0,

$$q(v) = \begin{cases} 0 & \text{if } v < \hat{v}, \\ 1 & \text{if } v \geq \hat{v} \end{cases} \quad \text{and} \quad t(v) = \begin{cases} t_0 & \text{if } v < \hat{v}, \\ t_0 + \hat{v} & \text{if } v \geq \hat{v}. \end{cases}$$

Individual rationality (IR) (seller cannot force buyer to participate):

$$q(v)v - t(v) \geq 0 \quad \text{for all } v.$$

Optimal deterministic direct selling mechanism: Seller solves

$$\max_{\hat{v}, t_0} F(\hat{v})t_0 + (1 - F(\hat{v}))(t_0 + \hat{v}) \quad \text{subject to IR.}$$

IR implies $t_0 = 0$.

$\Rightarrow$ Optimal allocation and payment is the same as in posted price mechanism!

1.3 Standard Auctions

Same setting as before, but with is a second potential buyer.

- Seller owns single indivisible object, valuation 0.
- Valuations v_1, v_2 of buyers 1 and 2 are independent realizations of V. Random variable V is continuously distributed according to F on $[0, 1]$.
- v_i is **privately** known to buyer i.

Seller could continue to use **posted price**...

- if both buyers want to buy, allocate randomly
- optimal posted price $p^* \in \arg\max_p \ (1 - F(p)^2)p$

...or use an **auction** instead.

Auction induces **Bayesian game** between buyers. Equilibrium notions:

- dominant strategy equilibrium
- Bayesian Nash equilibrium (less demanding)

Four standard auction formats

- Sealed-bid auctions (each bidder submits one bid in a sealed envelope)
  - **First-price auction**: Highest bidder wins and pays his bid.
  - **Second-price auction**: Highest bidder wins and pays the second-highest bid.
- Open auctions:
  - **English auction**: Seller starts with a price of zero and begins to increase it. Each bidder signals when he wishes to drop out. When only one bidder remains, he is the winner and pays the current price.
  - **Dutch auction**: Seller starts with a very high price and begins to reduce it. The first bidder to raise his hand wins and pays the current price.

Second-Price Auction

Payoff of bidder 1 (when bidder 2 bids b_2):

$$\Pi_1 = \begin{cases} v_1 - b_2 & \text{if } b_1 > b_2, \\ 0 & \text{if } b_1 < b_2. \end{cases}$$

Suppose bidder 2's bid is b_2 (unknown to bidder 1):

- Bidder 1 prefers to win the auction if $v_1 > b_2$.
- Bidder 1 prefers to lose the auction if $v_1 < b_2$.

Bidding $b_1 = v_1$ guarantees that bidder 1 wins exactly when he prefers to win!

$\Longrightarrow$ For bidder i, bidding $b_i = v_i$ is the unique (weakly) **dominant strategy**.

In the dominant strategy equilibrium of the second-price auction (SPA),

- the object is allocated *efficiently* (i.e., to the buyer with the highest value).
- the (interim) expected payment of buyer i with valuation v_i is

$$m^{SPA}(v_i) = F(v_i)E[V \,|V < v_i].$$

Note: there are other Bayesian Nash equilibria, with asymmetric strategies.

- Example: bidder 1 bids $b_1 = 1$ for all v_1, bidder 2 bids $b_2 = 0$ for all v_2. ($\rightarrow$ zero revenue for seller & potentially inefficient allocation)

Relation to **English auction**:

- In the English auction, it is a dominant strategy to drop out when the price reaches one's true valuation.
- Equivalent to the SPA: bidder with highest valuation wins and pays second highest valuation.

First-Price Auction

Payoff of bidder 1:

$$\Pi_1 = \begin{cases} v_1 - b_1 & \text{if } b_1 > b_2, \\ 0 & \text{if } b_1 < b_2. \end{cases}$$

In the first-price auction (FPA), there are no dominant strategies.

We will determine a symmetric Bayesian Nash equilibrium
where each bidder i bids $\beta(v_i)$ if his valuation is v_i.

- Suppose 2 uses the continuous and strictly increasing strategy $\beta(v_2)$.
- If bidder 1 with valuation v_1 bids b, his expected payoff is

$$\pi(v_1, b) := F(\beta^{-1}(b))(v_1 - b).$$

- Maximizing wrt b yields the FOC

$$f(\beta^{-1}(b))[\beta^{-1}(b)]'(v_1 - b) - F(\beta^{-1}(b)) = 0. \tag{1}$$

- Note $[\beta^{-1}(b)]' = \frac{1}{\beta'(\beta^{-1}(b))}$. In a symmetric equilibrium $b = \beta(v_1)$.

$$\frac{f(v_1)}{\beta'(v_1)}(v_1 - \beta(v_1)) - F(v_1) = 0 \tag{2}$$

- Rewriting the differential equation gives

$$v_1 f(v_1) = [F(v_1)\beta(v_1)]'.$$

- With initial condition $\beta(0) = 0$ (bidder with valuation 0 bids 0), solution is

$$\beta(v_1) = \frac{1}{F(v_1)} \int_0^{v_1} z f(z) dz.$$

Proposition

A symmetric Bayesian Nash equilibrium of the FPA is that both bidders bid

$$\beta(v_i) = \frac{1}{F(v_i)} \int_0^{v_i} z f(z) dz = E[V \,|V < v_i].$$

Proof.

Note that $\beta(v_i)$ is continuous and strictly increasing, as assumed at the outset. To complete the proof, we have to show that the FOC (1) is not only necessary but also sufficient for bidder 1's optimal bid.

- Suppose bidder 1 bids $b = \beta(z) \neq \beta(v_1)$. (1) implies

$$\pi_b'(v_1, \beta(z)) = \frac{f(z)}{\beta'(z)}(v_1 - \beta(z)) - F(z).$$

- From the differential equation (2), we know

$$-\frac{f(z)}{\beta'(z)}\beta(z) - F(z) = -\frac{f(z)}{\beta'(z)}z.$$

  Hence,

$$\pi_b'(v_1, \beta(z)) = \frac{f(z)}{\beta'(z)}(v_1 - z).$$

- Since β is strictly increasing, $\pi_b'(v_1, b) > (<)\ 0$ if $b < (>)\ \beta(v_1)$. □

In the symmetric Bayesian Nash equilibrium of the FPA,

- the object is allocated *efficiently*.
- the (interim) expected payment of buyer i with valuation v_i is

$$m^{FPA}(v_i) = F(v_i)\beta(v_i) = F(v_i)E[V \,|V < v_i].$$

It can be shown (not straightforward) that the symmetric equilibrium β is the unique Bayesian Nash equilibrium of the FPA.

Relation to **Dutch auction**:

- In the Dutch auction, each bidder chooses the price at which to buy. The bidder who chooses the highest price wins and pays this price.
- The Dutch auction is *strategically equivalent* to the FPA.
  $\Rightarrow$ Unique equilibrium: bidder i buys when price reaches $\beta(v_i)$.

Revenue Equivalence

Expected payment of a bidder with value v is the same in the FPA and the SPA:

$$m^{FPA}(v) = m^{SPA}(v) = m(v) := F(v)E[V\,|V < v]$$

$\Rightarrow$ Seller's expected revenue $2E[m(V)]$ is the same in all 4 standard auctions!

- Using mechanism design techniques, we will be able to explain revenue equivalence and show that it extends to many other auction formats.

- Expected revenue is the same, but ex post revenue typically differs.
- Revenue equivalence does in general not extend to settings with ex ante asymmetric bidders, interdependent values, or risk-averse bidders.

Optimal Auctions?

- Does the seller prefer standard auctions or a posted price?
- Can the seller do better than that?

Mechanism design theory will enable us to determine revenue maximizing mechanisms for more general auction settings.

For ex ante symmetric buyers, a combination of a standard auction with a posted price is optimal (i.e., an auction with a reserve price/minimum bid).

- In which settings does a standard auction result in an efficient allocation?

2 The Mechanism Design Problem

2.1 Environment

- n **agents** $i \in N := \{1, \ldots, n\}$
- set of possible **alternatives** X
- Each agent i has **private information** $\theta_i \in \Theta_i$. (θ_i is agent i's *type*.)
- Each agent i is an expected utility maximizer with vNM utility function

$$u_i(x, \theta) \quad \text{where } x \in X \text{ and } \theta \in \Theta := \Theta_1 \times \cdots \times \Theta_n.$$

- Type profile $\theta = (\theta_1, \ldots, \theta_n)$ is drawn from **commonly known distribution** with probability density $f(\cdot)$ over Θ.

Notation:

- $\Theta_{-i} := \Theta_1 \times \cdots \times \Theta_{i-1} \times \Theta_{i+1} \times \cdots \times \Theta_n$.
- For $\theta_i \in \Theta_i$ and $\theta_{-i} = (\theta_1, \ldots, \theta_{i-1}, \theta_{i+1}, \ldots, \theta_n) \in \Theta_{-i}$,

$$(\theta_i, \theta_{-i}) = (\theta_1, \ldots, \theta_{i-1}, \theta_i, \theta_{i+1}, \ldots, \theta_n).$$

Special case: Independent private values

Two often imposed assumptions:

1. **private values**:

$$u_i(x, \theta) = u_i(x, \theta_i) \quad \text{for all } i \in N \text{ and all } x \in X.$$

2. **independent types**: types are independently distributed, i.e., there are densities $f_i(\theta_i)$ such that

$$f(\theta) = \prod_{i \in N} f_i(\theta_i) \quad \text{for all } \theta \in \Theta.$$

In **independent private values environments** both of these assumptions hold.

Example 1: Public project with private values

E.g., building a bridge

- set of alternatives $X = \{0, 1\} \times \mathbb{R}^n$
- $x = (k, t_1, \ldots, t_n) \in X$:
  - if $k = 0$, bridge is not built; if $k = 1$, bridge is built
  - each agent i obtains monetary transfer t_i
- private information: $\theta_i \in \mathbb{R}$ is i's willingness to pay for the bridge.
- utility functions:

$$u_i(x, \theta) = u_i(x, \theta_i) = \theta_i k + t_i$$

Example 2: Matching

Marriage model from Part I:

- Agents: $N = M \cup W$
- Set of alternatives $X = \Psi$, the set of all possible matchings μ.
- private information about preferences: $\theta_i = P(i) \in \mathcal{P}_i = \Theta_i$.
- utility functions: some real-valued functions $u_i(\mu, \theta) = u_i(\mu, \theta_i)$ such that

$$u_i(\mu', \theta_i) > u_i(\mu, \theta_i) \quad \text{if and only if} \quad \mu'(i) \succ_i \mu(i) \text{ according to } P(i).$$

Example 3a: Auction without externalities

- Auction for one object, two bidders: $N = \{1, 2\}$, $X = \{0, 1, 2\} \times \mathbb{R}^2$
- $(k, t_1, t_2) \in X$:
  - if $k = 0$, object is not sold; if $k = i$, bidder i gets object
  - $-t_i$ is payment by bidder i
- private information: valuation for the object $\theta_i \in [0, 1]$.
- utility functions:

$$u_i((k, t_1, t_2), \theta) = \begin{cases} \theta_i + t_i & \text{if } k = i, \\ t_i & \text{if } k \neq i. \end{cases}$$

Example 3b: Auction with allocation externalities

- Environment as in example 3a, but with different types and utilities.
- private information: $\theta_i = (\theta_i^i, \theta_i^j) \in [0,1] \times [-1,0]$
- utility functions:

$$u_i((k,t_1,t_2),\theta) = \begin{cases} \theta_i^i + t_i & \text{if } k = i, \\ \theta_i^j + t_i & \text{if } k = j \neq i,\ j \neq 0, \\ t_i & \text{if } k = 0. \end{cases}$$

- *Example*: object is a license to use a cost reducing production technology in a duopoly. If the competitor of firm i obtains the license, i's profits are lower than if nobody obtains the license.

Example 4: Bilateral trade with interdependent values

- $N = \{1, 2\}$ where agent 1 is the owner of an object; $X = \{1, 2\} \times \mathbb{R}^2$.
- $x = (k, t_1, t_2) \in X$: if $k = 1$, agent 1 keeps object; if $k = 2$, agent 2 obtains object; t_i is a monetary transfer to agent i.

- Private information:
  - $\theta_1 = (q, v_1) \in [0, 1] \times [0, 1]$,
    where q is the quality of the object and v_1 is 1's taste for quality.
  - $\theta_2 = v_2 \in [0, 1]$, where v_2 is 2's taste for quality.

- utility functions:

$$u_1(x, \theta) = \begin{cases} qv_1 + t_1 & \text{if } k = 1, \\ t_1 & \text{if } k = 2, \end{cases} \qquad u_2(x, \theta) = \begin{cases} t_2 & \text{if } k = 1, \\ qv_2 + t_2 & \text{if } k = 2. \end{cases}$$

(2's utility depends on 1's private information)

2.2 Social Choice Functions and Mechanisms

Definition

A **social choice function** (SCF) is a function $c\colon \Theta \to X$
that, for each possible type profile θ, chooses an alternative $c(\theta) \in X$.

A desirable property of SCFs is ex post efficiency:

Definition

A SCF c is **ex post efficient** if there exists no $\theta \in \Theta$ such that for some $x \in X$

$$u_i(x, \theta) \geq u_i(c(\theta), \theta)\ \forall i \quad \text{and} \quad u_i(x, \theta) > u_i(c(\theta), \theta) \text{ for one } i.$$

Mechanisms

Collective choices are usually made indirectly through institutions in which agents interact. A mechanism is the formal representation such an institution.

Definition

A **mechanism** $\Gamma = (S_1, \ldots, S_n, g)$ consists of

- a **strategy set** S_i for each agent $i \in N$
- and an **outcome function** $g\colon S_1 \times \cdots \times S_n \to X$.

A mechanism defines the rules of a procedure for making a collective decision:

- S_i: allowed actions of each agent i
  (e.g., the bids in an auction; the allowable votes in an election)
- g: rule for how agents' actions are turned into a social choice
  (e.g., allocation and payments as a function of bids; set of elected candidates)

A mechanism need not be static.
(e.g., an auction/election may involve several rounds of bidding/voting)

The induced game of incomplete information

A mechanism Γ combined with the environment **induces** a **Bayesian game** $G_\Gamma := [N, \{S_i\}_{i \in N}, \{\tilde{u}_i\}_{i \in N}, \Theta, f(\cdot)]$ with payoffs

$$\tilde{u}_i(s_1, \ldots, s_n, \theta) := u_i(g(s_1, \ldots, s_n), \theta) \quad \forall (s_1, \ldots, s_n) \in S_1 \times \cdots \times S_n.$$

A strategy $s_i \colon \Theta_i \to S_i$ for agent i specifies a choice $s_i(\theta_i)$ for each type θ_i.

Two equilibrium concepts: A strategy profile $(s_1^*(\cdot), \ldots, s_n^*(\cdot))$ is

- a **dominant strategy equilibrium** if, for each $i \in N$ and $\theta \in \Theta$,

$$\tilde{u}_i(s_i^*(\theta_i), s_{-i}, \theta) \geq \tilde{u}_i(s_i', s_{-i}, \theta) \quad \forall s_i' \in S_i \text{ and } s_{-i} \in S_{-i}.$$

- a **Bayesian Nash equilibrium** if, for each $i \in N$ and $\theta_i \in \Theta_i$,

$$E_{\theta_{-i}}\left[\tilde{u}_i(s_i^*(\theta_i), s_{-i}^*(\theta_{-i}), \theta) \middle| \theta_i\right] \geq E_{\theta_{-i}}\left[\tilde{u}_i(\hat{s}_i, s_{-i}^*(\theta_{-i}), \theta) \middle| \theta_i\right] \quad \forall \hat{s}_i \in S_i.$$

Implementation

Definition

The mechanism $\Gamma = (S_1, \ldots, S_n, g)$ **implements** the SCF c if there is an equilibrium strategy profile $(s_1^*(\cdot), \ldots, s_n^*(\cdot))$ of the induced game G_Γ such that

$$g(s_1^*(\theta_1), \ldots, s_n^*(\theta_n)) = c(\theta) \quad \text{for all } \theta \in \Theta.$$

If Γ implements c, we say c is a **performance** of Γ.

- G_Γ may have several equilibria with different performances.
  We only require that the outcome of one equilibrium coincides with c.
- Depending on the equilibrium concept we use, we say
  - either: Γ implements c in dominant strategies; c is a dominant strategy performance of Γ.
  - or: Γ implements c in Bayesian Nash equilibrium; c is a Bayesian performance of Γ.

Example: Second-price auction

Single object is auctioned among n bidders using a second-price auction: the highest bidder wins, paying the second-highest bid.

Environment:

- Bidders $N = \{1, \ldots, n\}$, alternatives $X = \{0, 1, \ldots, n\} \times \mathbb{R}^n$
- $(k, t_1, \ldots, t_n) \in X$:
  - if $k = 0$, object is not sold; if $k = i$, bidder i gets object
  - $-t_i$ is payment by bidder i.
- types: $\Theta_i = [0, 1]$ for all i
- utility functions:

$$u_i((k, t_1, \ldots, t_n), \theta_i) = \begin{cases} \theta_i + t_i & \text{if } k = i, \\ t_i & \text{if } k \neq i. \end{cases}$$

Mechanism $\Gamma = (S_1, \ldots, S_n, g)$:

- For each i, the strategy set $S_i = \mathbb{R}_+$ is the set of possible bids.
- For each profile of bids $s = (s_1, \ldots, s_n) \in \mathbb{R}_+^n$,
  the outcome is $g(s) = (k(s), t_1(s), \ldots, t_n(s))$ where

$$k(s) = \min\{i \in N \mid s_i \geq s_j \; \forall j \in N\}$$

$$\text{and } t_i(s) = \begin{cases} -\max_{j \neq i} s_j & \text{if } k(s) = i \\ 0 & \text{otherwise} \end{cases}$$

Dominant strategy equilibrium: $s_i^*(\theta_i) = \theta_i$ for all i.

$c(\theta_1, \ldots, \theta_n) = g(s_1^*(\theta_1), \ldots, s_n^*(\theta_n)) = g(\theta_1, \ldots, \theta_n)$
is a dominant strategy performance of the second-price auction.

2.3 Direct Mechanisms and the Revelation Principle

Central Question: Which social choice functions are implementable in an environment with private information?

A SCF c depends on private information that agents cannot be forced to reveal.

But it may be possible to *design* a mechanism that implements the SCF c.
$\rightarrow$ A SCF is **implementable** if there exists a mechanism that implements it.

- Are ex post efficient SCFs implementable?
- Which implementable SCF maximizes a given objective?
  (e.g. expected welfare or utility of mechanism designer)

Commitment

We assume that the mechanism designer has full commitment power: he can set the rules of the mechanism and commit that he will not change the rules after the agents have chosen their actions.

Example: the mechanism designer is the seller in an auction

- The seller commits to refuse any renegotiation after the auction, e.g., if a non-winning bidder offers to pay more than the winner has to.
- In a second-price auction, the seller has to credibly commit to only charge the second-highest bid from the winner.

Sources of commitment:

- contracts
- reputation / repeated interaction

Direct Mechanisms

Problem: Set of possible mechanisms is very large.

A smaller class of mechanisms:

Definition

A mechanism $\Gamma = (S_1, \ldots, S_n, g)$ is a **direct mechanism** if $S_i = \Theta_i$ for all i.

Definition

A direct mechanism Γ is dominant strategy (Bayesian) **incentive compatible** if $(s_1^*(\cdot), \ldots, s_n^*(\cdot))$ with $s_i^*(\theta_i) = \theta_i$ for all $\theta_i \in \Theta_i$ and $i \in N$ is a dominant strategy (Bayesian Nash) equilibrium of the game G_Γ induced by Γ.

Truthful implementation

Definition

A SCF is **truthfully implementable** in dominant strategies (in Bayesian Nash equilibrium) if it is the performance of a dominant strategy (Bayesian) incentive compatible direct mechanism.

Remarks:

- Equivalent definition: c is truthfully implementable if $\Gamma = (\Theta_1, \ldots, \Theta_n, c)$ is an incentive compatible direct mechanism.
- A SCF that is truthfully implementable in dominant strategies is also called **strategy-proof**.

The Revelation Principle

Proposition

Let $\Gamma = (S_1, \ldots, S_n, g)$ *be any mechanism with dominant strategy (Bayesian) performance* c_Γ*. Then* $\Gamma' = (\Theta_1, \ldots, \Theta_n, c_\Gamma)$ *is a dominant strategy (Bayesian) incentive compatible direct mechanism.*

Corollary (Revelation Principle)

A SCF is implementable if and only if it is truthfully implementable.

We only prove the dominant-strategy version of the proposition.
(You will be asked to prove the Bayesian version as part of the next problem set.)

Proof.

Since c_Γ is a dominant strategy performance of Γ, there exist dominant strategies $s_1^*(\cdot), \dots, s_n^*(\cdot)$ in G_Γ such that

$$c_\Gamma(\theta) = g(s_1^*(\theta_1), \dots, s_n^*(\theta_n)).$$

Now consider Γ'. Fix agent i and the reports of the other agents $\hat{\theta}_{-i} \in \Theta_{-i}$. Payoff of agent i with type θ_i and report $\hat{\theta}_i$:

$$\begin{aligned} u_i(c_\Gamma(\hat{\theta}_i, \hat{\theta}_{-i}), \theta) &= u_i(g(s_i^*(\hat{\theta}_i), s_{-i}^*(\hat{\theta}_{-i})), \theta) \\ &\leq u_i(g(s_i^*(\theta_i), s_{-i}^*(\hat{\theta}_{-i})), \theta) = u_i(c_\Gamma(\theta_i, \hat{\theta}_{-i}), \theta) \end{aligned}$$

because $s_i^*(\theta_i)$ is a dominant strategy for player i in G_Γ. Hence, $\hat{\theta}_i = \theta_i$ is a dominant strategy for player i in $G_{\Gamma'}$ □

3 Quasi-Linear Private Values Environments

3.1 Setup

Throughout this section, we assume **quasi-linear** utilities and **private values**.

Each **alternative** $x = (k, t_1, \dots, t_n) \in X$ consists of

1. a physical **allocation** (or "project choice") $k \in K$,
2. a monetary **transfer** $t_i \in \mathbb{R}$ to each agent i.

Utility function of agent i:

$$u_i((k, t_1, \dots, t_n), \theta) = v_i(k, \theta_i) + t_i.$$

Remarks:

- $v_i(k, \theta_i)$ is the **value** of allocation k to agent i in terms of money.
- Agents are risk-neutral with respect to money (independent of wealth).
- Utility is freely *transferable* across agents.

Feasibility and Social Choice Functions

We assume that there is no outside source of financing (no budget deficit).
$\Rightarrow$ Transfers $t := (t_1, \ldots, t_n)$ are *feasible* if and only if $\sum_{i \in N} t_i \leq 0$.

Set of alternatives:

$$X = \Big\{ (k, t) \in K \times \mathbb{R}^n \;\Big|\; \sum_{i \in N} t_i \leq 0 \Big\}$$

A social choice function (SCF) $c = (k, t)$ consists of

an **allocation rule** $k\colon \Theta \to K$ and a **payment rule** $t\colon \Theta \to \mathbb{R}^n$

that assign an alternative $(k(\theta), t(\theta)) \in X$ to each type profile θ,
where $t(\theta) := (t_1(\theta), \ldots, t_n(\theta))$.

Ex post efficient SCFs

In a quasi-linear environment, a SCF $c = (k, t)$ is ex post efficient if and only if

- allocation rule k is **value maximizing**:

$$\sum_{i \in N} v_i(k(\theta), \theta_i) \geq \sum_{i \in N} v_i(\hat{k}, \theta_i) \quad \text{for all } \hat{k} \in K \text{ and } \theta \in \Theta,$$

- and payment rule t satisfies **budget balance**:

$$\sum_{i \in N} t_i(\theta) = 0 \quad \text{for all } \theta \in \Theta.$$

We next focus on dominant strategy implementation and study value maximization without and with budget balance. Then we turn to Bayesian implementation.

3.2 Value Maximization

Proposition

Let k^* be a value maximizing allocation rule. The SCF $c = (k^*, t)$ is truthfully implementable in dominant strategies if, for all $i \in N$,

$$t_i(\theta) = \sum_{j \neq i} v_j(k^*(\theta), \theta_j) + h_i(\theta_{-i}), \tag{3}$$

where h_i is an arbitrary function $h_i \colon \Theta_{-i} \to \mathbb{R}$.

Definition

A direct mechanism $\Gamma = (\Theta_1, \ldots, \Theta_n, c)$ with $c = (k^*, t)$ where k^* is value maximizing and t satisfies (3) is a **Vickrey-Clarke-Groves** (VCG) mechanism.

- named after Vickrey (1961), Clarke (1971), and Groves (1973)

Proof.

$c = (k^*, t)$ is truthfully implementable in dominant strategies if, for all $i \in N$, all $\theta_i, \hat{\theta}_i \in \Theta_i$, and all $\theta_{-i} \in \Theta_{-i}$,

$$v_i(k^*(\theta_i, \theta_{-i}), \theta_i) + t_i(\theta_i, \theta_{-i}) \geq v_i(k^*(\hat{\theta}_i, \theta_{-i}), \theta_i) + t_i(\hat{\theta}_i, \theta_{-i})$$
$$\iff \quad \sum_{j \in N} v_j(k^*(\theta_i, \theta_{-i}), \theta_j) \geq \sum_{j \in N} v_j(k^*(\hat{\theta}_i, \theta_{-i}), \theta_j).$$

This holds because $k^*(\theta_i, \theta_{-i})$ maximizes $\sum_{j \in N} v_j(k, \theta_j)$ for all $\theta \in \Theta$. □

Intuition: The transfer to an agent i who reports $\hat{\theta}_i$ consists of two parts.

1. $\sum_{j \neq i} v_j(k^*(\hat{\theta}_i, \theta_{-i}), \theta_j)$ is used to equate i's payoff with the total value. Hence, i's incentives are aligned with the goal of value maximization.
2. $h_i(\theta_{-i})$ does not distort incentives because it is independent of i's report.

The Pivot Mechanism

Let $k^*_{-i}(\theta_{-i})$ be an allocation rule that maximizes the value of all agents $j \neq i$:

$$\sum_{j \neq i} v_j(k^*_{-i}(\theta_{-i}), \theta_j) \geq \sum_{j \neq i} v_j(k, \theta_j) \quad \text{for all } k \in K \text{ and } \theta_{-i} \in \Theta_{-i}.$$

Definition (Clarke, 1971)

The **pivot mechanism** is a VCG mechanism with

$$h_i(\theta_{-i}) = -\sum_{j \neq i} v_j(k^*_{-i}(\theta_{-i}), \theta_j).$$

- In the pivot mechanism, the transfer to agent i is defined to be equal to the *externality* i imposes on the other agents:

$$t_i(\theta) = \sum_{j \neq i} \Big(v_j(k^*(\theta), \theta_j) - v_j(k^*_{-i}(\theta_{-i}), \theta_j) \Big)$$

Depending on the functions $h_i(\cdot)$, the payment rule t of a VCG mechanism may not be feasible and violate $\sum_{i \in N} t_i(\theta) \leq 0$.

Example: If $v_i(k, \theta_i) > 0 \; \forall i, k, \theta_i$, then $h_i(\theta_{-i}) = 0 \; \forall i$ leads to a budget deficit of $(n-1) \sum_{i \in N} v_i(k^*(\theta), \theta_i)$.

Proposition

The payment rule of the pivot mechanism satisfies $\sum_{i \in N} t_i(\theta) \leq 0$ *for all* $\theta \in \Theta$.

Proof.

$$\sum_{i \in N} t_i(\theta) = \sum_{i \in N} \left(\sum_{j \neq i} v_j(k^*(\theta), \theta_j) - \sum_{j \neq i} v_j(k^*_{-i}(\theta_{-i}), \theta_j) \right)$$

Since k^*_{-i} is value maximizing for the set of agents $j \neq i$,

$$\sum_{j \neq i} v_j(k^*_{-i}(\theta_{-i}), \theta_j) \geq \sum_{j \neq i} v_j(k^*(\theta), \theta_j) \quad \Longrightarrow \sum_{i \in I} t_i(\theta) \leq 0. \qquad \square$$

Example: Allocation of a single indivisible object

Environment: $N = \{1, \ldots, n\}$, $K = \{0, 1, \ldots, n\}$, $\Theta_i = [0, 1]$,

$$\text{and } v_i(k, \theta_i) = \begin{cases} \theta_i & \text{if } k = i, \\ 0 & \text{otherwise.} \end{cases}$$

Pivot mechanism (k^*, t):

$$k^*(\theta) = \min \left\{ i \in N \;\middle|\; \theta_i = \max_{j \in N} \theta_j \right\},$$

$$k^*_{-i}(\theta_{-i}) = \min \left\{ j \in N \setminus i \;\middle|\; \theta_j = \max_{l \in N \setminus i} \theta_l \right\},$$

$$t_i(\theta) = \sum\nolimits_{j \neq i} v_j(k^*(\theta), \theta_j) - \sum\nolimits_{j \neq i} v_j(k^*_{-i}(\theta_{-i}), \theta_j)$$

$$= \begin{cases} -\max_{l \in N \setminus i} \theta_l & \text{if } k^*(\theta) = i, \\ 0 & \text{otherwise.} \end{cases}$$

$\Rightarrow (k^*, t)$ is exactly the SCF that is implemented by the **second-price auction**!
(The SPA typically generates a budget surplus that goes to the seller.)

3.3 Budget Balance

Define

$$V(\theta) := \sum_{i \in N} v_i(k^*(\theta), \theta_i)$$

to be the total value generated by a value maximizing allocation rule k^*.

Proposition

A VCG mechanism that satisfies budget balance exists if and only if there are functions $V_1(\theta_{-1}), \ldots, V_n(\theta_{-n})$ such that

$$\sum_{i \in N} V_i(\theta_{-i}) = V(\theta).$$

Proof.

Suppose $h_1, \ldots, h_n$ define a VCG mechanism that satisfies budget balance, i.e., $\sum_{i \in N} t_i(\theta) = 0$. Then,

$$0 = \sum_{i \in N} \left(\sum_{j \neq i} v_j(k^*(\theta), \theta_j) + h_i(\theta_{-i}) \right)$$

$$\implies 0 = (n-1)V(\theta) + \sum_{i \in N} h_i(\theta_{-i})$$

$$\implies V(\theta) = -\frac{1}{n-1} \sum_{i \in N} h_i(\theta_{-i}).$$

Conversely, if $V(\theta) = \sum_{i \in N} V_i(\theta_{-i})$, we can set $h_i(\theta_{-i}) = -(n-1)V_i(\theta_{-i})$ to obtain a VCG mechanism with balanced budget. □

Example: public project

$N = \{1, 2\}$, $K = \{0, 1\}$, $\Theta_i = [-1, 1]$, and $v_i(k, \theta_i) = k\theta_i$.

$$\implies V(\theta) = \max\{0, \theta_1 + \theta_2\} \neq V_1(\theta_2) + V_2(\theta_1).$$

An important special case

Corollary

If $\Theta_i = \{\theta_i\}$, i.e., Θ_i is a singleton, for some $i \in N$, then there is an ex post efficient SCF that is truthfully implementable in dominant strategies.

Proof.

Set $V_i(\theta_{-i}) = V(\theta)$ and $V_j(\theta_{-j}) = 0$ for all $j \neq i$. □

- If $\Theta_i = \{\theta_i\}$, then agent i has no private information. Then we can use the pivot mechanism to provide incentives for the remaining agents $N \setminus i$ and transfer the budget surplus to agent i without distorting incentives.
- Example: second-price auction where the seller is an agent who has no private information.

Beyond VCG?

- Apart from VCG mechanisms, are there other direct mechanisms that implement a value maximizing allocation rule k^* in dominant strategies?
- The next result identifies a class of environments where this is not the case.

Let $\mathcal{V}$ denote the set of all possible functions $v\colon K \to \mathbb{R}$.

Proposition

*Suppose that for each agent $i \in N$, $\{v_i(\cdot, \theta_i) \mid \theta_i \in \Theta_i\} = \mathcal{V}$, i.e., every possible value function from K to $\mathbb{R}$ arises for some $\theta_i \in \Theta_i$. Then a SCF $c = (k^*, t)$ with value maximizing allocation rule k^* is truthfully implementable in dominant strategies if and **only if** t is the payment rule of a VCG mechanism.*

Proof

Consider SCF $c = (k^*, t)$ where k^* is value maximizing. For any payment rule t, define

$$h_i(\theta_i, \theta_{-i}) := t_i(\theta) - \sum_{j \neq i} v_j(k^*(\theta), \theta_j).$$

To show: $h_i(\theta_i, \theta_{-i})$ is independent of θ_i for all i if c is truthfully implementable.

Suppose *by contradiction* that c is truthfully implementable, but for some $\theta_i, \hat{\theta}_i$ and θ_{-i},

$$h_i(\theta_i, \theta_{-i}) > h_i(\hat{\theta}_i, \theta_{-i}).$$

Case 1: $k^*(\theta_i, \theta_{-i}) = k^*(\hat{\theta}_i, \theta_{-i})$. Then $t_i(\theta_i, \theta_{-i}) > t_i(\hat{\theta}_i, \theta_{-i})$ and therefore

$$v_i(k^*(\theta_i, \theta_{-i}), \hat{\theta}_i) + t_i(\theta_i, \theta_{-i}) > v_i(k^*(\hat{\theta}_i, \theta_{-i}), \hat{\theta}_i) + t_i(\hat{\theta}_i, \theta_{-i}).$$

$\Rightarrow$ contradiction! (c is not truthfully implementable.)

Case 2: $k^*(\theta_i, \theta_{-i}) \neq k^*(\hat{\theta}_i, \theta_{-i})$. Consider the type $\theta_i^\varepsilon \in \Theta_i$ for which, with $\varepsilon > 0$,

$$v_i(k, \theta_i^\varepsilon) = \begin{cases} -\sum_{j \neq i} v_j(k^*(\theta_i, \theta_{-i}), \theta_j) & \text{if } k = k^*(\theta_i, \theta_{-i}), \\ -\sum_{j \neq i} v_j(k^*(\hat{\theta}_i, \theta_{-i}), \theta_j) + \varepsilon & \text{if } k = k^*(\hat{\theta}_i, \theta_{-i}), \\ -\infty & \text{otherwise.} \end{cases}$$

$\implies k^*(\theta_i^\varepsilon, \theta_{-i}) = k^*(\hat{\theta}_i, \theta_{-i})$ since k^* is value maximizing.
$\implies h_i(\theta_i^\varepsilon, \theta_{-i}) = h_i(\hat{\theta}_i, \theta_{-i})$ by case 1.

By incentive compatibility for type θ_i^ε,

$$\begin{aligned} & v_i(k^*(\theta_i^\varepsilon, \theta_{-i}), \theta_i^\varepsilon) + t_i(\theta_i^\varepsilon, \theta_{-i}) \geq v_i(k^*(\theta_i, \theta_{-i}), \theta_i^\varepsilon) + t_i(\theta_i, \theta_{-i}) \\ \implies & v_i(k^*(\hat{\theta}_i, \theta_{-i}), \theta_i^\varepsilon) + t_i(\theta_i^\varepsilon, \theta_{-i}) \geq v_i(k^*(\theta_i, \theta_{-i}), \theta_i^\varepsilon) + t_i(\theta_i, \theta_{-i}) \\ \implies & \varepsilon + h_i(\theta_i^\varepsilon, \theta_{-i}) \geq h_i(\theta_i, \theta_{-i}) \\ \implies & \varepsilon + h_i(\hat{\theta}_i, \theta_{-i}) \geq h_i(\theta_i, \theta_{-i}) \end{aligned}$$

For small enough $\varepsilon > 0$, this contradicts $h_i(\theta_i, \theta_{-i}) > h_i(\hat{\theta}_i, \theta_{-i})$. $\square$

3.4 Ex post efficiency without dominant strategies

- As we have seen, there are environments where no ex post efficient SCF can be implemented in **dominant strategies**.
  - In many cases, VCG mechanisms cannot have a balanced budget.
  - But in some environments, VCG mechanisms are the only mechanisms that truthfully implement value maximizing allocations in dominant strategies.

- However, for *independent private values*, always at least one ex post efficient SCF can be implemented in **Bayesian Nash equilibrium**.

From now on, we assume statistically **independent types**,
i.e., for each agent i, θ_i is independently drawn from some distribution F_i.

The expected externality mechanism

Let k^* be a value maximizing allocation rule.

Define

$$\xi_i(\theta_i) := E_{\theta_{-i}}\Big[\sum_{j \neq i} v_j(k^*(\theta_i, \theta_{-i}), \theta_j)\Big].$$

- $\xi_i(\theta_i)$ represents the expected value to agents $j \neq i$ if i reports θ_i and all $j \neq i$ report truthfully. (Note: ξ_i is a function of only θ_i and *not* of θ_{-i}.)
- The change in ξ_i when agent i changes his report is the *expected externality* of this change on agents $j \neq i$.

Definition (d'Aspremont and Gérard-Varet, 1979; Arrow, 1979)

The **expected externality mechanism** is the direct mechanism $\Gamma = (\Theta_1, \ldots, \Theta_n, (k^*, t))$ where $t_i(\theta) = \xi_i(\theta_i) - \frac{1}{n-1} \sum_{j \neq i} \xi_j(\theta_j)$ for all i.

The following proposition implies that the expected externality mechanism truthfully implements an ex post efficient SCF.

Proposition

The SCF $c = (k^*, t)$ *is truthfully implementable in Bayesian Nash equilibrium if*

$$t_i(\theta) = \xi_i(\theta_i) + h_i(\theta_{-i}) \quad \text{for all } i \in N, \tag{4}$$

where h_i *is an arbitrary function* $h_i \colon \Theta_{-i} \to \mathbb{R}$.
The SCF $c = (k^*, t)$ *is ex post efficient if* t *satisfies (4) and*

$$h_i(\theta_{-i}) = -\frac{1}{n-1} \sum_{j \neq i} \xi_j(\theta_j).$$

Proof.

Consider agent i and suppose all agents $j \neq i$ report their types truthfully in the direct mechanism with outcome function $c = (k^*, t)$ where t satisfies (4). i's expected payoff if he has type θ_i and reports $\hat{\theta}_i$ is

$$E_{\theta_{-i}}\Big[v_i(k^*(\hat{\theta}_i, \theta_{-i}), \theta_i) + t(\hat{\theta}_i, \theta_{-i})\Big]$$

$$= E_{\theta_{-i}}\Big[\sum_{j \in N} v_j(k^*(\hat{\theta}_i, \theta_{-i}), \theta_j)\Big] + E_{\theta_{-i}}[h_i(\theta_{-i})].$$

The first part is maximized at report $\hat{\theta}_i = \theta_i$ since $k = k^*(\theta)$ maximizes $\sum_{j \in N} v_j(k, \theta_j)$. The second part, $E_{\theta_{-i}}[h_i(\theta_{-i})]$, is independent of report $\hat{\theta}_i$.
$\implies \hat{\theta}_i = \theta_i$ is best response of i, i.e., c is truthfully implementable in BNE.

If $h_i(\theta_{-i}) = -\frac{1}{n-1} \sum_{j \neq i} \xi_j(\theta_j)$, t satisfies budget balance since

$$\sum_{i \in N} t_i(\theta) = \sum_{i \in N} \xi_i(\theta_i) - \frac{1}{n-1} \sum_{i \in N} \sum_{j \neq i} \xi_j(\theta_j) = 0.$$

□

Remarks

- Weakening the equilibrium concept to Bayesian Nash equilibrium makes implementation of an ex post efficient SCF possible in general.

  Drawback of Bayesian implementation: payment rule t of the expected externality mechanism depends on type distributions $F_1, \ldots, F_n$.

- The expected externality mechanism implements one specific ex post efficient SCF. $\rightarrow$ It results in a particular distribution of utility across agents.

- What other SCFs are implementable in Bayesian Nash equilibrium?
  - There may be additional requirements (e.g. *participation constraints*) that the expected externality mechanism does not satisfy.
  - We may be interested in goals other than ex post efficiency, e.g., maximizing the utility of one agent / revenue.

In the next section, we identify the class of SCFs that are implementable in Bayesian Nash equilibrium if utility functions are linear in types.

4 Bayesian Implementation with Linear Utility

4.1 Characterization of Bayesian implementable SCFs

In this section, we focus on **linear** utilities and **independent private values**.

Assumptions:

- Set of possible types of agent i is an interval:

$$\Theta_i = [\underline{\theta}_i, \overline{\theta}_i] \subset \mathbb{R}, \quad \text{with } \underline{\theta}_i < \overline{\theta}_i.$$

- θ_i is independently drawn from a distribution with probability density f_i satisfying $f_i(\theta_i) > 0$ for all $\theta_i \in [\underline{\theta}_i, \overline{\theta}_i]$.

- Utility function of agent i:

$$u_i((k,t),\theta_i) = \theta_i v_i(k) + t_i.$$

Definitions

Consider a SCF $c = (k, t)$ and suppose all agents $j \neq i$ truthfully report their types θ_{-i} in the direct mechanism $\Gamma = (\Theta_1, \ldots, \Theta_n, c)$.

Define for agent i who reports $\hat{\theta}_i$

- the interim **expected allocation**

$$\overline{v}_i(\hat{\theta}_i) := E_{\theta_{-i}}\left[v_i(k(\hat{\theta}_i, \theta_{-i}))\right]$$

- and the interim **expected transfer**

$$\overline{t}_i(\hat{\theta}_i) := E_{\theta_{-i}}\left[t_i(\hat{\theta}_i, \theta_{-i})\right].$$

If all agents report truthfully, agent i's interim **expected utility** given his type θ_i is

$$U_i(\theta_i) := E_{\theta_{-i}}\left[u_i((k(\theta), t(\theta)), \theta_i)\right] = \theta_i \overline{v}_i(\theta_i) + \overline{t}_i(\theta_i).$$

Bayesian Incentive Compatibility

A direct mechanism with SCF $c = (k, t)$ is **Bayesian incentive compatible** if and only if truthtelling is a Bayesian Nash equilibrium, i.e.,

$$U_i(\theta_i) \geq \theta_i \overline{v}_i(\hat{\theta}_i) + \overline{t}_i(\hat{\theta}_i) \quad \text{for all } \theta_i, \hat{\theta}_i \in \Theta_i \text{ and } i \in N. \tag{5}$$

Proposition

The direct mechanism with SCF $c = (k, t)$ is Bayesian incentive compatible ***if and only if****, for all $i \in N$,*

1. $\overline{v}_i(\theta_i)$ *is non-decreasing in θ_i,* (IC1)
2. $U_i(\theta_i) = U_i(\underline{\theta}_i) + \int_{\underline{\theta}_i}^{\theta_i} \overline{v}_i(z)dz$ *for all $\theta_i \in \Theta_i$.* (IC2)

Proof: necessity ("only if")

Suppose c is incentive compatible. Then (5) implies, for $\hat{\theta}_i > \theta_i$,

$$U_i(\theta_i) \geq \theta_i \overline{v}_i(\hat{\theta}_i) + \overline{t}_i(\hat{\theta}_i) = U_i(\hat{\theta}_i) + (\theta_i - \hat{\theta}_i)\overline{v}_i(\hat{\theta}_i)$$

and $U_i(\hat{\theta}_i) \geq \hat{\theta}_i \overline{v}_i(\theta_i) + \overline{t}_i(\theta_i) = U_i(\theta_i) + (\hat{\theta}_i - \theta_i)\overline{v}_i(\theta_i)$.

$$\implies \overline{v}_i(\hat{\theta}_i) \geq \frac{U_i(\hat{\theta}_i) - U_i(\theta_i)}{\hat{\theta}_i - \theta_i} \geq \overline{v}_i(\theta_i). \qquad (6)$$

$\implies \overline{v}_i(\theta_i)$ is non-decreasing in θ_i.

According to (5), U_i is the maximum of a family of affine functions.

$\implies U_i$ is a convex function. $\implies U_i$ is absolutely continuous.

$\implies U_i$ is differentiable almost everywhere
and equal to the integral of its derivative.

Letting $\hat{\theta}_i \to \theta_i$ in (6) yields $U_i'(\theta_i) = \overline{v}_i(\theta_i)$ for almost all θ_i

and therefore $U_i(\theta_i) - U_i(\underline{\theta}_i) = \int_{\underline{\theta}_i}^{\theta_i} \overline{v}_i(z)dz.$ □

Proof: sufficiency ("if")

Consider any θ_i and $\hat{\theta}_i$ and suppose (IC1) and (IC2) hold. Hence,

$$U_i(\theta_i) - U_i(\hat{\theta}_i) = \int_{\hat{\theta}_i}^{\theta_i} \overline{v}_i(z)dz.$$

If $\theta_i > \hat{\theta}_i$,

$$\begin{aligned} U_i(\theta_i) &= U_i(\hat{\theta}_i) + \int_{\hat{\theta}_i}^{\theta_i} \overline{v}_i(z)dz \\ &\geq U_i(\hat{\theta}_i) + \int_{\hat{\theta}_i}^{\theta_i} \overline{v}_i(\hat{\theta}_i)dz = U_i(\hat{\theta}_i) + (\theta_i - \hat{\theta}_i)\overline{v}_i(\hat{\theta}_i) = \theta_i\overline{v}_i(\hat{\theta}_i) + \overline{t}_i(\hat{\theta}_i). \end{aligned}$$

If $\theta_i < \hat{\theta}_i$,

$$\begin{aligned} U_i(\theta_i) &= U_i(\hat{\theta}_i) - \int_{\theta_i}^{\hat{\theta}_i} \overline{v}_i(z)dz \\ &\geq U_i(\hat{\theta}_i) - \int_{\theta_i}^{\hat{\theta}_i} \overline{v}_i(\hat{\theta}_i)dz = U_i(\hat{\theta}_i) - (\hat{\theta}_i - \theta_i)\overline{v}_i(\hat{\theta}_i) = \theta_i\overline{v}_i(\hat{\theta}_i) + \overline{t}_i(\hat{\theta}_i). \end{aligned}$$

Therefore, c is incentive compatible as (5) is satisfied for all θ_i and $\hat{\theta}_i$. □

Implications for SCFs

The proposition identifies the class of SCFs that are Bayesian implementable:

- The allocation rule k has to satisfy (IC1): each $\overline{v}_i$ has to be non-decreasing. (This requirement cannot be relaxed through the payment rule.)

- Given such an allocation rule k, (IC2) pins down interim expected transfers up to a constant ($\overline{t}_i(\underline{\theta}_i)$):

$$U_i(\theta_i) = U_i(\underline{\theta}_i) + \int_{\underline{\theta}_i}^{\theta_i} \overline{v}_i(z)dz$$

$$\implies \overline{t}_i(\theta_i) = \overline{t}_i(\underline{\theta}_i) + \underline{\theta}_i \overline{v}_i(\underline{\theta}_i) - \theta_i \overline{v}_i(\theta_i) + \int_{\underline{\theta}_i}^{\theta_i} \overline{v}_i(z)dz \quad \text{for all } i \text{ and } \theta_i.$$

- Given k and $\overline{t}_i(\underline{\theta}_i)$, the payment rule t has to be such that

$$E_{\theta_{-i}}[t_i(\theta_i, \theta_{-i})] = \overline{t}_i(\theta_i) \quad \text{for all } i \text{ and } \theta_i.$$

(Typically, there are many such t.)

Payoff Equivalence

Consider any two direct mechanisms that Bayesian implement the same allocation rule k.

Incentive compatibility implies that the interim expected utility given θ_i

$$U_i(\theta_i) = U_i(\underline{\theta}_i) + \int_{\underline{\theta}_i}^{\theta_i} \overline{v}_i(z)dz$$

of each agent i is the **same** in both mechanisms **except** for the **constant** $U_i(\underline{\theta}_i)$.

4.2 Application: Revenue Equivalence of Standard Auctions

Auction environment with *symmetric* independent private values:

- single indivisible object, n bidders
- Bidder i has valuation $\theta_i \in [\underline{\theta}, \overline{\theta}]$ for the object.
- Each θ_i is independently drawn from the same distribution ($f_i = f\ \forall i$).
- Set of possible allocations: $K = \{1, 2, \ldots, n\}$ where $k = i$ denotes that bidder i obtains the object.
- Bidder i's payoff from alternative (k, t):

$$u_i((k,t), \theta_i) = \theta_i v_i(k) + t_i \quad \text{with } v_i(k) = \begin{cases} 1 & \text{if } k = i, \\ 0 & \text{otherwise.} \end{cases}$$

A standard auction

Consider an (anonymous) auction that

1. assigns the object to the **highest bidder**,
2. has a **symmetric** Bayesian Nash equilibrium
   with strictly **increasing** bidding strategy $\beta \colon [\underline{\theta}, \overline{\theta}] \to \mathbb{R}_+$.

$\Longrightarrow$ in equilibrium, bidder i with the highest valuation θ_i obtains object.

$\Longrightarrow$ auction implements a SCF (k^*, t), where k^* is value maximizing,
i.e., $k^*(\theta) = i$ if $\theta_i > \max_{j \neq i} \theta_j$.

By the *Revelation Principle*, the direct mechanism with SCF (k^*, t) is Bayesian incentive compatible. $\Longrightarrow$ (IC2) holds:

$$\theta_i \overline{v}_i(\theta_i) + \overline{t}_i(\theta_i) = \underline{\theta}\overline{v}_i(\underline{\theta}) + \overline{t}_i(\underline{\theta}) + \int_{\underline{\theta}}^{\theta_i} \overline{v}_i(z) dz$$

Since $\overline{v}_i(\theta_i) = \Pr_{\theta_{-i}}[\max_{j \neq i} \theta_j < \theta_i] = F(\theta_i)^{n-1}$, (IC2) implies

$$\bar{t}_i(\theta_i) = -\theta_i F(\theta_i)^{n-1} + \int_{\underline{\theta}}^{\theta_i} F(z)^{n-1} dz + \bar{t}_i(\underline{\theta}) \qquad (7)$$

where $\bar{t}_i(\underline{\theta}) = \bar{t}_j(\underline{\theta})$ for all i, j because the auction equilibrium is symmetric.

Seller's (ex ante) **expected revenue**:

$$\begin{aligned}\sum_{i \in N} E_\theta[-t_i(\theta)] &= \sum_{i \in N} E_{\theta_i}\big[-\bar{t}_i(\theta_i)\big] \\ &= -n E_{\theta_i}\big[\bar{t}_i(\theta_i)\big] \\ &= n \int_{\underline{\theta}}^{\overline{\theta}} \left(y F(y)^{n-1} - \int_{\underline{\theta}}^{y} F(z)^{n-1} dz \right) f(y) dy - n \bar{t}_i(\underline{\theta}).\end{aligned}$$

Revenue Equivalence

Corollary (Revenue Equivalence Theorem)

Consider any two auctions that award the object to the highest bid and have a symmetric Bayesian Nash equilibrium with strictly increasing bidding strategy.

If the interim expected payment by a bidder with valuation $\underline{\theta}$ is the same in this equilibrium of both auctions, then

- *the **interim expected payment** from each type of each bidder*
- *and therefore the **expected revenue** for the seller*

are the same in both auctions.

Example: Second-price auction

- We have shown that bidding one's valuation is a dominant strategy.
  $\Rightarrow$ The SPA has a symmetric BNE with strictly increasing $\beta_S(\theta_i) = \theta_i$.

- The interim expected payment by type θ_i is

$$-\bar{t}_i(\theta_i) = \Pr_{\theta_{-i}}\left[\max_{j\neq i}\theta_j < \theta_i\right] E_{\theta_{-i}}\left[\max_{j\neq i}\theta_j \,\middle|\, \max_{j\neq i}\theta_j < \theta_i\right]$$
$$= \int_{\underline{\theta}}^{\theta_i} z g(z) dz$$

  where $G(z) := F(z)^{n-1}$ is the probability distribution of $z = \max_{j\neq i}\theta_j$.

- Applying *integration by parts*, we obtain

$$-\bar{t}_i(\theta_i) = \theta_i G(\theta_i) - \int_{\underline{\theta}}^{\theta_i} G(z) dz.$$

  This is consistent with equation (7) implied by (IC2).
  ($\bar{t}_i(\underline{\theta}) = 0$ since in the SPA only the winner makes a payment.)

Example: First-price auction

We can use Revenue Equivalence to determine the symmetric Bayesian Nash equilibrium of the FPA with n bidders and non-uniformly distributed valuations.

- Suppose the FPA has a strictly increasing symmetric equilibrium strategy $\beta_F(\theta_i)$. Like in the SPA, only the winner makes a payment, hence the interim expected payment by type $\underline{\theta}$ is zero.

- Revenue Equivalence: $-\bar{t}_i(\theta_i)$ is the same in the SPA and the FPA.

$$-\bar{t}_i^S(\theta_i) = \Pr_{\theta_{-i}}\Big[\max_{j\neq i}\theta_j < \theta_i\Big] E_{\theta_{-i}}\Big[\max_{j\neq i}\theta_j \Big| \max_{j\neq i}\theta_j < \theta_i\Big]$$

$$-\bar{t}_i^F(\theta_i) = \Pr_{\theta_{-i}}\Big[\max_{j\neq i}\theta_j < \theta_i\Big]\beta_F(\theta_i)$$

$$\implies \beta_F(\theta_i) = E_{\theta_{-i}}\Big[\max_{j\neq i}\theta_j \Big| \max_{j\neq i}\theta_j < \theta_i\Big].$$

Let us verify that this is a Bayesian Nash equilibrium of the FPA:

- The equilibrium strategy

$$\beta_F(\theta_i) = E_{\theta_{-i}}\left[\max_{j\neq i}\theta_j \,\middle|\, \max_{j\neq i}\theta_j < \theta_i\right] = \frac{1}{G(\theta_i)}\int_{\underline{\theta}}^{\theta_i} zg(z)dz$$

  is indeed strictly increasing.

- Deviations to bids $b \notin [\beta_F(\underline{\theta}), \beta_F(\overline{\theta})]$ are not profitable for bidder i.

- Deviations to bids $b \in [\beta_F(\underline{\theta}), \beta_F(\overline{\theta})]$ are equivalent to bidder i with valuation θ_i bidding $\beta_F(\tilde{\theta}_i)$ instead of $\beta_F(\theta_i)$. This is not profitable since for all θ_i and $\tilde{\theta}_i$

$$\begin{aligned} & G(\theta_i)(\theta_i - \beta_F(\theta_i)) \geq G(\tilde{\theta}_i)(\theta_i - \beta_F(\tilde{\theta}_i)) \\ \Leftrightarrow \quad & (G(\theta_i) - G(\tilde{\theta}_i))\theta_i \geq \int_{\tilde{\theta}_i}^{\theta_i} zg(z)dz \\ \Leftrightarrow \quad & \int_{\tilde{\theta}_i}^{\theta_i} (\theta_i - z)g(z)dz \geq 0. \end{aligned}$$

4.3 Participation Constraints

- Up to this point, we have implicitly assumed that agents can be forced to participate in a mechanism.
  - Agents could choose optimal actions from those allowed by the mechanism
  - but agents could not choose whether or not to participate.

- In many application, however, participation is *voluntary*.
  - e.g. a seller owning an object may not be forced to trade it (property rights)

Suppose each agent i has an outside option:

Agent i can obtain utility $\hat{u}_i(\theta_i)$ by withdrawing from the mechanism.

A mechanism is **individually rational** for agent i if it is optimal for i to participate.

Types of Individual Rationality (IR)

Depending on the stage at which an agent makes the participation decision, we distinguish three types of individual rationality.

A direct mechanism with SCF $c = (k, t)$ satisfies for agent i

- **ex post individual rationality** if

$$u_i((k(\theta), t(\theta)), \theta_i) \geq \hat{u}_i(\theta_i) \quad \text{for all } \theta \in \Theta.$$

- **interim individual rationality** if

$$U_i(\theta_i) = E_{\theta_{-i}}\big[u_i((k(\theta), t(\theta)), \theta_i)\big] \geq \hat{u}_i(\theta_i) \quad \text{for all } \theta_i \in \Theta_i.$$

- **ex ante individual rationality** if

$$E[U_i(\theta_i)] = E_\theta\big[u_i((k(\theta), t(\theta)), \theta_i)\big] \geq E[\hat{u}_i(\theta_i)].$$

Remarks

Note that

$$\text{ex post IR} \quad \Longrightarrow \quad \text{interim IR} \quad \Longrightarrow \quad \text{ex ante IR}$$

but the reverse is not true.

- Ex post IR imposes the most severe constraint, ex ante IR the least severe.
- Which type of individual rationality constraint is relevant, depends on the particular application we study.

In general, private information may restrict the set of implementable SCFs not only through incentive compatibility constraints, but also through individual rationality constraints.

Example: Public project

- $N = \{1, 2\}$, $K = \{0, 1\}$, $\Theta_i = \{\underline{\theta}, \overline{\theta}\}$ where $0 < 2\underline{\theta} < \overline{\theta}$.
- The project costs $c \in (2\underline{\theta}, \overline{\theta})$; agent i pays share $\alpha_i \geq 0$ ($\alpha_1 + \alpha_2 = 1$).
- Utility function: $u_i((k, t), \theta_i) = k(\theta_i - \alpha_i c) + t_i$.
- Value maximizing allocation: $k^*(\theta) = \begin{cases} 0 & \text{if } \theta_1 = \theta_2 = \underline{\theta}, \\ 1 & \text{otherwise.} \end{cases}$

Suppose $\hat{u}_i(\theta_i) = 0$ for all θ_i and i. Is there an **ex post individually rational** direct mechanism that implements k^* in *dominant strategies*?

- Ex post IR for $\theta_1 = \underline{\theta}$ requires $\quad \underline{\theta} - \alpha_1 c + t_1(\underline{\theta}, \overline{\theta}) \geq 0$.
- IC for $\theta_1 = \overline{\theta}$ requires $\quad \overline{\theta} - \alpha_1 c + t_1(\overline{\theta}, \overline{\theta}) \geq \overline{\theta} - \alpha_1 c + t_1(\underline{\theta}, \overline{\theta})$.

  Hence, $t_1(\overline{\theta}, \overline{\theta}) \geq -\underline{\theta} + \alpha_1 c$. Similarly, $t_2(\overline{\theta}, \overline{\theta}) \geq -\underline{\theta} + \alpha_2 c$.
- Such transfers are **not feasible**: $t_1(\overline{\theta}, \overline{\theta}) + t_2(\overline{\theta}, \overline{\theta}) \geq -2\underline{\theta} + c > 0$.

5 Optimal Auctions

(Myerson, 1981)

5.1 Preliminaries

Consider a seller with a single object who faces several potential buyers.

The seller has many options when choosing the selling procedure:

- posted price
- (non-)standard auction
- negotiate with one or several buyers
- ...

We are now in a position to answer the question from Section 1:

- What kind of mechanisms are *optimal* (revenue maximizing) for the seller?

- Put differently: Which implementable SCFs maximize the seller's utility?

Environment

- Single indivisible object.
- Agent 0: seller with valuation $\theta_0 = 0$ (no private information).
- Agent $i \in N := \{1, 2, \ldots, n\}$: buyer with valuation $\theta_i \in [\underline{\theta}_i, \overline{\theta}_i]$, $\overline{\theta}_i > 0$.
- Each θ_i is independently drawn from a continuous distribution F_i with density $f_i(\theta_i) > 0$ for all $\theta_i \in [\underline{\theta}_i, \overline{\theta}_i]$.
- Set of possible allocations:

$$K = \left\{ (q_1, \ldots, q_n) \in [0,1]^n \;\middle|\; \sum_{i \in N} q_i \leq 1 \right\}.$$

  - Buyer i obtains the object with probability q_i.
  - With probability $q_0 := 1 - \sum_{i \in N} q_i$ the seller keeps the object.
- Utility functions: $u_i((k,t), \theta_i) = \theta_i q_i + t_i$ (i.e., linear with $v_i(k) = q_i$).

Restriction to **budget balanced transfers**: $t_0 = -\sum_{i \in N} t_i$.

- Wlog: if alternative with $t_0 < -\sum_{i \in N} t_i$ is implementable, then also $\tilde{t}_0 = -\sum_{i \in N} t_i$ is implementable and makes seller strictly better off.

Outside option: $\hat{u}_i(\theta_i) = 0$ for all i and $\theta_i \in [\underline{\theta}_i, \overline{\theta}_i]$.

We require **interim individual rationality**.

- A bidder in an auction mechanism decides whether to participate while knowing his valuation. By submitting a bid, he commits to accept the auction outcome.

Social choice function: For all $\theta \in [\underline{\theta}_1, \overline{\theta}_1] \times \cdots \times [\underline{\theta}_n, \overline{\theta}_n]$,

- allocation rule $q(\theta) := (q_1(\theta), \ldots, q_n(\theta)) \in K$,
- payment rule $t(\theta) := (t_1(\theta), \ldots, t_n(\theta)) \in \mathbb{R}^n$
  and by budget balance, $t_0(\theta) = -\sum_{i \in N} t_i(\theta)$.

Maximizing the seller's expected revenue

Revelation principle: restrict attention to Bayesian incentive compatible direct mechanisms with SCF (q, t).

Seller's expected revenue: $E[t_0(\theta)] = E[-\sum_{i \in N} t_i(\theta)]$.

Constrained optimization problem:

$$\max_{(q,t)} E\Big[-\sum_{i \in N} t_i(\theta)\Big]$$ subject to

- Bayesian incentive compatibility
- interim individual rationality

Incentive compatibility and individual rationality

Recall the notation from Section 4:

- $\overline{q}_i(\theta_i) := \overline{v}_i(\theta_i) = E_{\theta_{-i}}[q_i(\theta_i, \theta_{-i})]$ and $\overline{t}_i(\theta_i) = E_{\theta_{-i}}[t_i(\theta_i, \theta_{-i})]$,
- $U_i(\theta_i) = \theta_i \overline{q}_i(\theta_i) + \overline{t}_i(\theta_i)$.

Bayesian incentive compatibility is equivalent to (IC1) and (IC2), i.e.,

- $\overline{q}_i(\theta_i)$ is non-decreasing,
- $U_i(\theta_i) = U_i(\underline{\theta}_i) + \displaystyle\int_{\underline{\theta}_i}^{\theta_i} \overline{q}_i(z)dz.$

$$\iff \quad -\overline{t}_i(\theta_i) = -U_i(\underline{\theta}_i) + \theta_i \overline{q}_i(\theta_i) - \int_{\underline{\theta}_i}^{\theta_i} \overline{q}_i(z)dz.$$

Interim individual rationality requires $U_i(\theta_i) \geq 0$ for all $\theta_i \in [\underline{\theta}_i, \overline{\theta}_i]$.

(IC2) and $\overline{q}_i(\cdot) \geq 0$ implies $U_i(\theta_i) \geq U_i(\underline{\theta}_i)$ for all θ_i.

$\Rightarrow$ Incentive compatible mechanisms are individually rational if $U_i(\underline{\theta}_i) \geq 0$.

The optimization problem

Note that $E[-\sum_{i\in N} t_i(\theta)] = \sum_{i\in N} E[-t_i(\theta)] = \sum_{i\in N} E[-\bar{t}_i(\theta_i)]$.

The constrained optimization problem can be written as

$$\max_{(q,t)} \sum_{i\in N} E\left[-\bar{t}_i(\theta_i)\right]$$

subject to

1. $\bar{q}_i(\theta_i)$ is non-decreasing,
2. $-\bar{t}_i(\theta_i) = -U_i(\underline{\theta}_i) + \theta_i \bar{q}_i(\theta_i) - \int_{\underline{\theta}_i}^{\theta_i} \bar{q}_i(z)dz$,
3. $U_i(\underline{\theta}_i) \geq 0$.

$\rightarrow$ We use the 2nd constraint to substitute for $-\bar{t}_i(\theta_i)$ in the objective function.

Ex ante expected payments

$$E[-\bar{t}_i(\theta_i)] = -\int_{\underline{\theta}_i}^{\overline{\theta}_i} \bar{t}_i(\theta_i) f_i(\theta_i) d\theta_i$$
$$= -U_i(\underline{\theta}_i) + \int_{\underline{\theta}_i}^{\overline{\theta}_i} \theta_i \overline{q}_i(\theta_i) f_i(\theta_i) d\theta_i - \int_{\underline{\theta}_i}^{\overline{\theta}_i} \int_{\underline{\theta}_i}^{\theta_i} \overline{q}_i(z) dz \; f_i(\theta_i) d\theta_i.$$

Integration by parts yields

$$\int_{\underline{\theta}_i}^{\overline{\theta}_i} \int_{\underline{\theta}_i}^{\theta_i} \overline{q}_i(z) dz \; f_i(\theta_i) d\theta_i = \Big[\int_{\underline{\theta}_i}^{\theta_i} \overline{q}_i(z) dz \; F_i(\theta_i) \Big]_{\underline{\theta}_i}^{\overline{\theta}_i} - \int_{\underline{\theta}_i}^{\overline{\theta}_i} \overline{q}_i(\theta_i) F_i(\theta_i) d\theta_i$$
$$= \int_{\underline{\theta}_i}^{\overline{\theta}_i} \overline{q}_i(\theta_i) \big(1 - F_i(\theta_i)\big) d\theta_i.$$

Hence,

$$E[-\bar{t}_i(\theta_i)] = -U_i(\underline{\theta}_i) + \int_{\underline{\theta}_i}^{\overline{\theta}_i} \overline{q}_i(\theta_i) \left(\theta_i - \frac{1 - F_i(\theta_i)}{f_i(\theta_i)} \right) f_i(\theta_i) d\theta_i.$$

Definition

The **virtual valuation** of a buyer i with valuation θ_i is

$$\psi_i(\theta_i) := \theta_i - \frac{1 - F_i(\theta_i)}{f_i(\theta_i)}.$$

$$\begin{aligned} E[-\bar{t}_i(\theta_i)] &= -U_i(\underline{\theta}_i) + \int_{\underline{\theta}_i}^{\bar{\theta}_i} \bar{q}_i(\theta_i)\psi_i(\theta_i) f_i(\theta_i) d\theta_i \\ &= -U_i(\underline{\theta}_i) + E_{\theta_i}\left[\bar{q}_i(\theta_i)\psi_i(\theta_i)\right] = -U_i(\underline{\theta}_i) + E_\theta\left[q_i(\theta)\psi_i(\theta_i)\right]. \end{aligned}$$

Expected revenue:

$$\begin{aligned} \sum_{i \in N} E\left[-\bar{t}_i(\theta_i)\right] &= \sum_{i \in N} E_\theta\left[q_i(\theta)\psi_i(\theta_i)\right] - \sum_{i \in N} U_i(\underline{\theta}_i) \\ &= E_\theta\left[\sum_{i \in N} q_i(\theta)\psi_i(\theta_i)\right] - \sum_{i \in N} U_i(\underline{\theta}_i) \end{aligned}$$

5.2 Revenue Maximizing Mechanisms

Revenue maximizing direct mechanisms (q, t) solve

$$\max_{\substack{q,\\ \bar{t}_1(\underline{\theta}_1),\ldots,\bar{t}_n(\underline{\theta}_n)}} E_\theta\left[\sum_{i\in N} q_i(\theta)\psi_i(\theta_i)\right] - \sum_{i\in N} U_i(\underline{\theta}_i)$$

s.t.

1. $\overline{q}_i(\theta_i)$ is non-decreasing,
2. $U_i(\underline{\theta}_i) \geq 0$.

- The 2nd constraint is binding: optimal payment rules t are such that

$$U_i(\underline{\theta}_i) = 0 \quad \Longleftrightarrow \quad -\bar{t}_i(\underline{\theta}_i) = \underline{\theta}_i \overline{q}_i(\underline{\theta}_i).$$

- The problem reduces to choosing the allocation rule q to maximize the first part of the objective subject to $\overline{q}_i(\theta_i)$ being non-decreasing.

Regularity and pointwise maximum

Regularity assumption: For all $i \in N$, the distribution F_i is such that

$$\psi_i(\theta_i) \text{ is strictly increasing in } \theta_i. \tag{R}$$

Remark: (R) is satisfied if the *hazard rate* $\frac{f_i(\theta_i)}{1-F_i(\theta_i)}$ is non-decreasing. This is true for many distributions, e.g., the uniform, normal, logistic, chi-squared, exponential, or Laplace distribution.

- Ignoring the monotonicity constraint on $\overline{q}_i(\theta_i)$, the objective function is maximized if q maximizes $\sum_{i \in N} q_i(\theta)\psi_i(\theta_i)$ for every possible θ, i.e.,

$$q_i(\theta) = \begin{cases} 1 & \text{if } \psi_i(\theta_i) > \max\left\{0, \max_{j \neq i} \psi_j(\theta_j)\right\}, \\ 0 & \text{if } \psi_i(\theta_i) < \max\left\{0, \max_{j \neq i} \psi_j(\theta_j)\right\}. \end{cases}$$

- (R) implies $\psi_i(\theta_i) > \psi_i(\tilde{\theta}_i)$ whenever $\theta_i > \tilde{\theta}_i$. Hence, for all $\theta_i > \tilde{\theta}_i$,

$$q_i(\theta_i, \theta_{-i}) \geq q_i(\tilde{\theta}_i, \theta_{-i}) \quad \text{and therefore} \quad \overline{q}_i(\theta_i) \geq \overline{q}_i(\tilde{\theta}_i).$$

- Under (R), the solution to the unconstrained problem satisfies the monotonicity constraint and thus also solves to the constrained problem!

- The resulting revenue $E_\theta\left[\sum_{i \in N} q_i(\theta)\psi_i(\theta_i)\right]$ is the expectation of highest virtual valuation, provided it is non-negative.

Without assumption (R), solving for the revenue maximizing mechanism requires an additional step, which is sometimes called *ironing*. Details can be found in Myerson (1981).

The main result

Proposition

Assume (R). A mechanism maximizes the seller's expected revenue if and only if it Bayesian implements (q^*, t) *such that for all* $i \in N$ *and* $\theta_i \in [\underline{\theta}_i, \overline{\theta}_i]$

$$q_i^*(\theta) = \begin{cases} 1 & \text{if } \psi_i(\theta_i) > \max\{0, \max_{j \neq i} \psi_j(\theta_j)\}, \\ 0 & \text{if } \psi_i(\theta_i) < \max\{0, \max_{j \neq i} \psi_j(\theta_j)\} \end{cases} \tag{8}$$

and

$$-\bar{t}_i(\theta_i) = \theta_i \overline{q}_i^*(\theta_i) - \int_{\underline{\theta}_i}^{\theta_i} \overline{q}_i^*(z) dz. \tag{9}$$

The resulting expected revenue is

$$E_\theta\Big[\max\{\psi_1(\theta_1), \psi_2(\theta_2), \dots, \psi_n(\theta_n), 0\}\Big].$$

Implementation in dominant strategies

There are optimal transfers that induce even a dominant strategy equilibrium.

Define $y_i(\theta_{-i}) := \inf \left\{ \theta_i \in [\underline{\theta}_i, \overline{\theta}_i] \;\middle|\; \psi_i(\theta_i) \geq 0 \text{ and } \psi_i(\theta_i) \geq \psi_j(\theta_j) \, \forall j \neq i \right\}$.

- *Infimum*: greatest lower bound of a set; for the empty set, $\inf(\varnothing) = \infty$.
- Given θ_{-i}, $y_i(\theta_{-i})$ is the lowest type θ_i such that $q_i^*(\theta) = 1$.

Proposition

Assume (R). The direct mechanism with allocation rule q^* and payment rule

$$-t_i(\theta) = \begin{cases} y_i(\theta_{-i}) & \text{if } q_i^*(\theta) = 1, \\ 0 & \text{if } q_i^*(\theta) = 0. \end{cases}$$

$\forall i \in N$ is ***dominant strategy*** *incentive compatible and revenue maximizing.*

This mechanism is also **ex post** individually rational.

Proof.

The optimal allocation rule q^* given in (8) is equivalent to

$$q_i^*(\theta) = \begin{cases} 1 & \text{if } \theta_i > y_i(\theta_{-i}), \\ 0 & \text{if } \theta_i < y_i(\theta_{-i}). \end{cases}$$

Define the payment rule as

$$-t_i(\theta) = \theta_i q_i^*(\theta) - \int_{\underline{\theta}_i}^{\theta_i} q_i^*(z, \theta_{-i}) dz.$$

This is an optimal payment rule since taking the expectation $E_{\theta_{-i}}[\cdot]$ gives (9).

$$-t_i(\theta) = \begin{cases} \theta_i - \int_{y_i(\theta_{-i})}^{\theta_i} 1 dz = y_i(\theta_{-i}) & \text{if } \theta_i > y_i(\theta_{-i}), \\ 0 & \text{if } \theta_i < y_i(\theta_{-i}). \end{cases}$$

The utility of buyer i with valuation θ_i who reports $\tilde{\theta}_i$ is

$$\theta_i q_i^*(\tilde{\theta}_i, \theta_{-i}) + t_i(\tilde{\theta}_i, \theta_{-i}) = \begin{cases} \theta_i - y_i(\theta_{-i}) & \text{if } \tilde{\theta}_i > y_i(\theta_{-i}), \\ 0 & \text{if } \tilde{\theta}_i < y_i(\theta_{-i}). \end{cases}$$

$\Rightarrow$ Reporting $\tilde{\theta}_i = \theta_i$ maximizes i's utility for all θ_{-i}. □

5.3 Discussion and Implications

The interim expected payment from type θ_i of buyer i is

$$-\bar{t}_i(\theta_i) = \underbrace{\theta_i \overline{q}_i^*(\theta_i)}_{i\text{'s value (surplus)}} - \underbrace{\int_{\underline{\theta}_i}^{\theta_i} \overline{q}_i^*(z)dz.}_{i\text{'s information rent}}$$

- Private information prevents the seller from extracting all the surplus.
- Tradeoff between surplus extraction and efficiency:
  - The efficient (i.e., value maximizing) allocation rule maximizes total surplus
  - but by choosing an inefficient allocation rule the seller may reduce the information rent of the buyers.
- Tradeoff is resolved by choosing q^* that maximizes $E\left[\sum_{i\in N} \psi_i(\theta_i)q_i(\theta)\right]$ instead of the efficient q that maximizes $E\left[\sum_{i\in N} \theta_i q_i(\theta)\right]$.

Virtual valuations and inefficiencies

Interpretation of $\psi_i(\theta_i)$:

$$E[\psi_i(\theta_i)q_i(\theta)] = \int_{\underline{\theta}_i}^{\overline{\theta}_i} \overline{q}_i(\theta_i)\big(\theta_i f_i(\theta_i) - (1 - F_i(\theta_i))\big)d\theta_i$$

- An increase in $\overline{q}_i(\theta_i)$ leads to
  - $\theta_i f_i(\theta_i)$: an increase in surplus of type θ_i which occurs with prob. $f_i(\theta_i)$.
  - $-(1 - F_i(\theta_i))$: a decrease in the payments from types $\tilde{\theta}_i > \theta_i$. Payments from $\tilde{\theta}_i$ must be decreased as deviating to report θ_i became more attractive.

Allocating the object based on ψ_i typically leads to **two kinds of inefficiencies**:

1. Suppose $\max_i \theta_i > 0$, but $\max_i \psi_i(\theta_i) < 0$.

   In a revenue-maximizing mechanism, the seller keeps the object although there is a buyer with a strictly positive valuation for it.

2. Suppose $F_i \neq F_j$ and therefore $\psi_j \neq \psi_i$.

   If $\psi_i(\theta_i) > \psi_j(\theta_j) \geq 0$ but $\theta_i < \theta_j$, a revenue maximizing mechanism allocates the object to i although j has a higher valuation for it.

The Symmetric Case

Suppose buyers are ex ante symmetric: $\underline{\theta}_i = \underline{\theta}$, $\overline{\theta}_i = \overline{\theta}$ and $F_i = F$ for all i.

- Hence, $\psi_i = \psi$ for all i.
- Assuming (R), q^* allocates to the highest θ_i, provided that $\psi(\theta_i) > 0$.
- Define

$$r^* := \begin{cases} \psi^{-1}(0) & \text{if } \psi(\underline{\theta}) < 0, \\ 0 & \text{if } \psi(\underline{\theta}) \geq 0. \end{cases}$$

  Hence, $y_i(\theta_{-i}) = \max\{r^*, \max_{j \neq i} \theta_j\}$.

Proposition

Suppose (R) and bidders are ex ante symmetric. Then a ***second-price auction*** *with* ***reserve price*** r^* *is a revenue maximizing mechanism.*

- SPA with reserve price r: if the highest bid is below r, seller keeps object; otherwise, winner pays maximum of r and the second highest bid.
- If $\underline{\theta} \geq \frac{1}{f(\underline{\theta})}$, $r^* = 0$. $\Rightarrow$ standard second-price auction, ex post efficiency

The One Buyer Case

Suppose there is only one buyer ($n = 1$) and $\underline{\theta}_1 = 0$.

Under assumption (R), the revenue maximizing mechanism as defined in (8) and (9) yields allocation and payment rule

$$q_1^*(\theta) = \begin{cases} 1 & \text{if } \theta_1 > \psi^{-1}(0), \\ 0 & \text{if } \theta_1 < \psi^{-1}(0) \end{cases} \quad \text{and} \quad -\bar{t}_1(\theta_1) = \begin{cases} \psi^{-1}(0) & \text{if } \theta_1 > \psi^{-1}(0), \\ 0 & \text{if } \theta_1 < \psi^{-1}(0). \end{cases}$$

This is equivalent to the allocation and payment implemented when the seller uses the **optimal posted price** p^* of Section 1.2:

$$p^* = \arg\max_p \ (1 - F(p))p \quad \Longleftrightarrow \quad p^* = \psi^{-1}(0)$$

Two Asymmetric Buyers

Suppose there are two buyers with $\underline{\theta}_1 = \underline{\theta}_2 = 0$, $\overline{\theta}_1 = \overline{\theta}_2 = \overline{\theta}$, but $F_1 \neq F_2$.
Assume that F_1 *dominates* F_2 in terms of the *hazard rate*:

$$\frac{f_1(x)}{1 - F_1(x)} < \frac{f_2(x)}{1 - F_2(x)} \quad \text{for all } x \in (0, \overline{\theta}).$$

- One can show: $\Rightarrow$ first order stochastic dominance $F_1(x) < F_2(x)$.
- The *strong* buyer 1 has a reputation for being more interested in the object than the *weak* buyer 2.

The optimal auction distorts the allocation **in favor of** the **weak** buyer:

- As $\psi_1(x) < \psi_2(x)\ \forall x \in (0, \overline{\theta})$, 1 wins only if $\theta_1 > \psi_1^{-1}(\psi_2(\theta_2)) > \theta_2$.
- Types $\theta_i < r_i$ never get object, where $r_1 := \psi_1^{-1}(0) > \psi_2^{-1}(0) =: r_2$. More types of 1 are excluded from trade (*discriminatory* reserve prices).
- Handicapping stronger buyer enhances competition (reduces info. rents).

5.4 Standard Auctions vs. Optimal Auctions

Advantages of standard auctions like the FPA or SPA without reserve price:

1. **anonymous**: rules are independent of identities of the bidders (fairness…).
2. **detail-free**: rules are universal, independent of details of the environment.

Optimal auctions are not detail-free:

- Allocation and payment rule depend on the type distributions $F_1, \dots, F_n$.
- Seller needs detailed knowledge about environment to specify the rules.
- Even in symmetric case, reserve price of optimal SPA depends on details.

$\rightarrow$ How much does the seller lose when using a standard auction instead of an optimal mechanism?

The value of competition

- We will in the following focus on symmetric buyers, i.e., $F_i = F$ for all i.

Proposition (Bulow and Klemperer, 1996)

Suppose (R) and buyers are ex ante symmetric. Then, the revenue from an optimal mechanism with n buyers is not larger than the revenue from the standard second-price auction (without reserve price) with $n+1$ buyers.

- The revenue of an optimal auction is bounded by the revenue of a standard auction with one additional bidder.
- Instead of worrying about optimal reserve price (optimally negotiating), seller is better off by inviting an additional buyer.

Proof (Kirkegaard, 2006)

Step 1. Let $O_n := E\left[\max\{\psi(\theta_1), \ldots, \psi(\theta_n), 0\}\right]$ denote the revenue in an optimal mechanism with n buyers.

Suppose there is an additional constraint: the seller cannot the keep object.

- Set of possible allocations: $\hat{K} := \{(q_1, \ldots, q_n) \in [0,1]^n \mid \sum_{i \in N} q_i = 1\}$.
- Let C_n denote the revenue in an optimal mechanism with n buyers when the set of possible allocations is constrained to $\hat{K} \subset K$.

Lemma

$O_n \leq C_{n+1}$.

Proof.

If there are $n+1$ buyers and the seller cannot keep the object, the seller can run an optimal mechanism among buyers $1, \ldots, n$ and allocate the object to buyer $n+1$ instead of keeping it, i.e., set $q_{n+1} = 1 - \sum_{i=1}^{n} q_i$. This yields revenue O_n, which is therefore a lower bound for C_{n+1}. □

Step 2.

To find the constrained optimal mechanism when the seller cannot keep the object, we follow exactly the same steps as for the optimal mechanism:

- Allocation rule q has to be monotone and maximize $E[\sum_{i \in N} q_i(\theta)\psi_i(\theta_i)]$.
- Solution with $q(\theta) \in \hat{K}$: always allocate to the buyer with highest $\psi_i(\theta_i)$.
- $\implies C_n = E[\max\{\psi(\theta_1), \ldots, \psi(\theta_n)\}]$

With symmetric bidders:

- constrained optimal mechanisms are ex post efficient,
- interim expected payment from type $\underline{\theta}$ is zero (binding IR constraint).

$\implies$ The standard SPA is a constrained optimal mechanism (revenue equiv.).

$\implies$ The standard SPA with $n+1$ bidders yields revenue $C_{n+1} \geq O_n$. □

6 Bilateral Trade

(Myerson and Satterthwaite, 1983)

6.1 Setup

One seller S who owns a single object, one potential buyer B.

- S has valuation for (or cost of producing) the object $\theta_S \in [\underline{\theta}_S, \overline{\theta}_S]$.
- B has valuation $\theta_B \in [\underline{\theta}_B, \overline{\theta}_B]$.
- Each θ_i is independently drawn from a continuous distribution with density $f_i(\theta_i) > 0$ for all $\theta_i \in [\underline{\theta}_i, \overline{\theta}_i]$.
- $\underline{\theta}_S < \overline{\theta}_B$ and $\underline{\theta}_B < \overline{\theta}_S$.

Set of possible allocations $K = [0, 1]$.

- With probability $p \in K$ there is trade: object is transferred from S to B.

Utility functions:

$$u_S((p,t),\theta_S) = t_S - p\theta_S \qquad \text{(i.e., } v_S(k) = -k \text{),}$$
$$u_B((p,t),\theta_B) = p\theta_B + t_B \qquad \text{(i.e., } v_B(k) = k \text{).}$$

Social choice functions, ex post efficiency

A SCF consists of

- allocation rule $p\colon [\underline{\theta}_S, \overline{\theta}_S] \times [\underline{\theta}_B, \overline{\theta}_B] \to [0,1]$,
- payment rule $t = (t_S, t_B)$, where $t_i\colon [\underline{\theta}_S, \overline{\theta}_S] \times [\underline{\theta}_B, \overline{\theta}_B] \to \mathbb{R}$.

A SCF (p, t) is **ex post efficient** if

- p is a **value** (gains from trade) **maximizing** allocation rule

$$p^*(\cdot) \in \arg\max_{p(\cdot)} (\theta_B - \theta_S) p(\theta_S, \theta_B).$$

$$\implies p^*(\theta_S, \theta_B) = \begin{cases} 1 & \text{if } \theta_S < \theta_B, \\ 0 & \text{if } \theta_S > \theta_B. \end{cases}$$

- transfers satisfy **budget balance**: $t_S(\theta_S, \theta_B) = -t_B(\theta_S, \theta_B)$.

Incentive compatibility and individual rationality

Let $\overline{p}_S(\theta_S) := E_{\theta_B}[p(\theta_S, \theta_B)]$ and $\overline{p}_B(\theta_B) := E_{\theta_S}[p(\theta_S, \theta_B)]$.

Hence, $U_S(\theta_S) = \overline{t}_S(\theta_S) - \overline{p}_S(\theta_S)\theta_S$ and $U_B(\theta_B) = \overline{p}_B(\theta_B)\theta_B + \overline{t}_B(\theta_B)$.

Bayesian incentive compatibility is equivalent to (IC1) and (IC2).

1. (IC1): $\overline{p}_S(\theta_S)$ is *non-increasing* and $\overline{p}_B(\theta_B)$ is *non-decreasing*.
2. (IC2): $$U_S(\theta_S) = U_S(\underline{\theta}_S) - \int_{\underline{\theta}_S}^{\theta_S} \overline{p}_S(z)dz = U_S(\overline{\theta}_S) + \int_{\theta_S}^{\overline{\theta}_S} \overline{p}_S(z)dz,$$
$$U_B(\theta_B) = U_B(\underline{\theta}_B) + \int_{\underline{\theta}_B}^{\theta_B} \overline{p}_B(z)dz.$$

Interim individual rationality: For $i = S, B$, $U_i(\theta_i) \geq \hat{u}_i(\theta_i) = 0$ for all θ_i.

- (IC1) & (IC2) imply $U_S'(\theta_S) \leq 0$ and $U_B'(\theta_B) \geq 0$.
- Hence, Bayesian IC mechanisms are interim IR if

$$U_S(\overline{\theta}_S) \geq 0 \quad \text{and} \quad U_B(\underline{\theta}_B) \geq 0.$$

6.2 The Myerson–Satterthwaite Theorem

Proposition (Myerson–Satterthwaite Theorem)

For the environment described in 6.1, there is no Bayesian incentive compatible direct mechanism with ex post efficient SCF that is interim individually rational for the seller and the buyer.

Proof

Consider the VCG mechanism with

$$h_S(\theta_B) = \min\{0, \overline{\theta}_S - \theta_B\} \quad \text{and} \quad h_B(\theta_S) = -\max\{0, \underline{\theta}_B - \theta_S\}.$$

This dominant strategy implements allocation rule $p^*(\theta_S, \theta_B)$ with payment rule

$$t_S(\theta_S, \theta_B) = \begin{cases} \min\{\theta_B, \overline{\theta}_S\} & \text{if } \theta_S < \theta_B, \\ 0 & \text{if } \theta_S > \theta_B, \end{cases}$$

$$t_B(\theta_S, \theta_B) = \begin{cases} -\max\{\theta_S, \underline{\theta}_B\} & \text{if } \theta_S < \theta_B, \\ 0 & \text{if } \theta_S > \theta_B. \end{cases}$$

$\implies U_S(\overline{\theta}_S) = 0$ and $U_B(\underline{\theta}_B) = 0$. $\implies$ Interim IR is satisfied.

However, budget balance is not satisfied: If $\theta_S < \theta_B$,

$$t_S(\theta_S, \theta_B) + t_B(\theta_S, \theta_B) = \min\{\theta_B, \overline{\theta}_S\} - \max\{\theta_S, \underline{\theta}_B\} > 0.$$

Let $\tilde{U}_S(\theta_S)$ and $\tilde{U}_B(\theta_B)$ denote the interim utilities in the VCG mechanism.

- Because of (IC2), the interim utilities in every Bayesian IC direct mechanism with allocation rule p^* satisfy

$$U_i(\theta_i) = \tilde{U}_i(\theta_i) + D_i \qquad \text{for some constants } D_S \text{ and } D_B.$$

  Since $\tilde{U}_S(\overline{\theta}_S) = \tilde{U}_B(\underline{\theta}_B) = 0$, interim IR requires $D_S \geq 0$ and $D_B \geq 0$.

- Budget balance implies $E[t_S(\theta_S, \theta_B) + t_B(\theta_S, \theta_B)] = 0$.

- Note that $\bar{t}_i(\theta_i) = E_{\theta_{-i}}[\tilde{t}_i(\theta_S, \theta_B)] + D_i$ for $i = S, B$.
  For every IC and IR mechanism that implements p^*,

$$\begin{aligned} E[t_S(\theta_S, \theta_B) + t_B(\theta_S, \theta_B)] &= E[\bar{t}_S(\theta_S) + \bar{t}_B(\theta_B)] \\ &= E[\tilde{t}_S(\theta_S, \theta_B) + \tilde{t}_B(\theta_S, \theta_B)] + D_S + D_B > 0. \end{aligned}$$

  Hence, no Bayesian IC and interim IR direct mechanism with SCF (p^*, t) satisfies budget balance. □

Discussion

- The proof reveals that the impossibility result still holds if we only require the budget to balance *in expectation* instead of ex post.

- According to the *Coase Theorem*, under complete information and if there are no transactions costs, the initial allocation of property rights is irrelevant for an ex post efficient outcome, because agents will continue to engage in transactions as long as the allocation remains inefficient.
- The Myerson-Satterthwaite Theorem shows that this is not valid under private information:
  There is no voluntary trading institution or bargaining process that ensures ex post efficient reallocation of the seller's property right to the object.

If we depart from the assumptions in 6.1, the impossibility result need not hold.

- Suppose the intervals $[\underline{\theta}_S, \overline{\theta}_S]$ and $[\underline{\theta}_B, \overline{\theta}_B]$ do not overlap:
  - If $\overline{\theta}_B < \underline{\theta}_S$, no trade/payments is ex post efficient and implementable.
  - If $\overline{\theta}_S < \underline{\theta}_B$, it is efficient to always trade. $p^*(\theta_S, \theta_B) = 1$ can be implemented with a fixed price $t_S(\theta_S, \theta_B) = -t_B(\theta_S, \theta_B) = x \in [\overline{\theta}_S, \underline{\theta}_B]$.

- Suppose types are drawn from the following discrete distributions:
  - $\theta_S \in \{1, 4\}$, $\theta_B \in \{0, 3\}$; each type has probability $\frac{1}{2}$.
  - Allocation p^* with transfers $t_S(\theta_S, \theta_B) = -t_B(\theta_S, \theta_B) = 2$ if $\theta_S < \theta_B$ and $t_S(\theta_S, \theta_B) = -t_B(\theta_S, \theta_B) = 0$ otherwise is IC and interim IR.

- Suppose we require only ex ante IR instead of interim IR:
  - The *expected externality mechanism* is ex post efficient and Bayesian IC.
  - With an additional fixed payment from the buyer to the seller we can transfer ex ante expected utility between agents and achieve ex ante IR.

6.3 Maximizing the Expected Gains from Trade

- The Myerson-Satterthwaite Theorem shows that the *first-best* outcome is unattainable: there is no mechanism that realizes all gains from trade.
- What is the *second-best* outcome that can be achieved under the incentive compatibility, individual rationality, and budget balance constraints?
- We will determine the mechanisms that maximize the sum of the ex ante expected utilities of the buyer and the seller (welfare).

  Under budget balance, this is equivalent to maximizing the ex ante expected gains from trade (expected value).

The optimization problem

$$\max_{(p,t)} \; E\big[p(\theta_S,\theta_B)(\theta_B-\theta_S)\big]$$

subject to

- budget balance: $t_S(\theta_S,\theta_B) = -t_B(\theta_S,\theta_B)$.
- Bayesian IC:
  - $\overline{p}_S(\theta_S)$ is non-increasing and $\overline{p}_B(\theta_B)$ is non-decreasing,
  - $U_S(\theta_S) = U_S(\overline{\theta}_S) + \int_{\theta_S}^{\overline{\theta}_S} \overline{p}_S(z)dz, \quad U_B(\theta_B) = U_B(\underline{\theta}_B) + \int_{\underline{\theta}_B}^{\theta_B} \overline{p}_B(z)dz.$
- interim IR: $U_S(\overline{\theta}_S) \geq 0$ and $U_B(\underline{\theta}_B) \geq 0$.

Weighted virtual valuation and weighted virtual cost

Definition

Let $\alpha \in [0,1]$. The α**-weighted virtual cost** of the seller with type θ_S is

$$\phi_S^\alpha(\theta_S) := \theta_S + \alpha \frac{F_S(\theta_S)}{f_S(\theta_S)}.$$

The α**-weighted virtual valuation** of the buyer with valuation θ_B is

$$\psi_B^\alpha(\theta_B) := \theta_B - \alpha \frac{1 - F_B(\theta_B)}{f_B(\theta_B)}.$$

Regularity assumption: The distributions F_S and F_B are such that

$$\phi_S^1(\theta_S) \text{ and } \psi_B^1(\theta_B) \text{ are strictly increasing.} \qquad \text{(R2)}$$

Sufficient for (R2): non-increasing *reverse hazard rate* $\frac{f_S}{F_S}$ and non-decreasing *hazard rate* $\frac{f_B}{1-F_B}$.

Ex ante expected transfers

For the **seller**, (IC2) implies

$$\bar{t}_S(\theta_S) = U_S(\overline{\theta}_S) + \overline{p}_S(\theta_S)\theta_S + \int_{\theta_S}^{\overline{\theta}_S} \overline{p}_S(z)dz,$$

$$E[\bar{t}_S(\theta_S)] = U_S(\overline{\theta}_S) + \int_{\underline{\theta}_S}^{\overline{\theta}_S} \overline{p}_S(\theta_S)\theta_S f_S(\theta_S)d\theta_S$$

$$+ \int_{\underline{\theta}_S}^{\overline{\theta}_S} \int_{\theta_S}^{\overline{\theta}_S} \overline{p}_S(z)dz f_S(\theta_S)d\theta_S.$$

Integration by parts yields

$$\int_{\underline{\theta}_S}^{\overline{\theta}_S} \int_{\theta_S}^{\overline{\theta}_S} \overline{p}_S(z)dz f_S(\theta_S)d\theta_S = \int_{\underline{\theta}_S}^{\overline{\theta}_S} \overline{p}_S(\theta_S)F_S(\theta_S)d\theta_S.$$

Hence,

$$\begin{aligned} E[\bar{t}_S(\theta_S)] &= U_S(\overline{\theta}_S) + \int_{\underline{\theta}_S}^{\overline{\theta}_S} \overline{p}_S(\theta_S)\left(\theta_S + \frac{F_S(\theta_S)}{f_S(\theta_S)}\right) f_S(\theta_S)d\theta_S \\ &= U_S(\overline{\theta}_S) + E[\overline{p}_S(\theta_S)\phi_S^1(\theta_S)]. \end{aligned}$$

For the **buyer**, (IC2) has the same implication as in Section 5 (slide 86):

$$\bar{t}_B(\theta_B) = U_B(\underline{\theta}_B) - \overline{p}_B(\theta_B)\theta_B + \int_{\underline{\theta}_B}^{\theta_B} \overline{p}_B(z)dz,$$

$$E[\bar{t}_B(\theta_B)] = U_B(\underline{\theta}_B) - \int_{\underline{\theta}_B}^{\overline{\theta}_B} \overline{p}_B(\theta_B)\left(\theta_B - \frac{1 - F_B(\theta_B)}{f_B(\theta_B)}\right) f_B(\theta_B)d\theta_B$$

$$= U_B(\underline{\theta}_B) - E\left[\overline{p}_B(\theta_B)\psi_B^1(\theta_B)\right].$$

Sum of ex ante expected transfers:

$$E\left[\bar{t}_S(\theta_S) + \bar{t}_B(\theta_B)\right] = U_S(\overline{\theta}_S) + E\left[\overline{p}_S(\theta_S)\phi_S^1(\theta_S)\right] + U_B(\underline{\theta}_B) - E\left[\overline{p}_B(\theta_B)\psi_B^1(\theta_B)\right]$$

$$= U_S(\overline{\theta}_S) + U_B(\underline{\theta}_B) - E\left[p(\theta_S, \theta_B)\left(\psi_B^1(\theta_B) - \phi_S^1(\theta_S)\right)\right]$$

Under **budget balance**

$$t_S(\theta_S, \theta_B) + t_B(\theta_S, \theta_B) = 0 \quad \Rightarrow \quad E\big[t_S(\theta_S, \theta_B) + t_B(\theta_S, \theta_B)\big] = 0,$$

and therefore (IC2) implies

$$U_S(\overline{\theta}_S) + U_B(\underline{\theta}_B) - E\big[p(\theta_S, \theta_B)\big(\psi_B^1(\theta_B) - \phi_S^1(\theta_S)\big)\big] = 0. \qquad (10)$$

Lemma

Consider an allocation rule p *such that* $\overline{p}_S(\theta_S)$ *is non-increasing and* $\overline{p}_B(\theta_B)$ *is non-decreasing. There is a budget balanced payment rule* t *that makes the direct mechanism with SCF* (p, t) *Bayesian IC and interim IR* ***if and only if***

$$E\big[p(\theta_S, \theta_B)\big(\psi_B^1(\theta_B) - \phi_S^1(\theta_S)\big)\big] \geq 0. \qquad (11)$$

Proof

1. only if:

- If the direct mechanism satisfies (IC2) and budget balance, (10) holds.
- Interim IR implies $U_S(\overline{\theta}_S) + U_B(\underline{\theta}_B) \geq 0$ and therefore (11).

2. if:

Suppose (11) holds. Then we can construct a budget balanced payment rule t that satisfies all conditions as follows (there are also other possibilities): Set

$$t_S(\theta_S, \theta_B) = -t_B(\theta_S, \theta_B)$$
$$= \overline{p}_S(\theta_S)\theta_S + \int_{\theta_S}^{\overline{\theta}_S} \overline{p}_S(z)dz - E[\overline{p}_B(\theta_B)\psi_B^1(\theta_B)]$$
$$+ \overline{p}_B(\theta_B)\theta_B - \int_{\underline{\theta}_B}^{\theta_B} \overline{p}_B(z)dz.$$

For the seller,

$$\bar{t}_S(\theta_S) = E_{\theta_B}\left[t_S(\theta_S,\theta_B)\right] = \overline{p}_S(\theta_S)\theta_S + \int_{\theta_S}^{\overline{\theta}_S} \overline{p}_S(z)dz.$$

Hence, (IC2) is satisfied with $U_S(\overline{\theta}_S) = 0.$ Therefore interim IR is also satisfied.

For the buyer,

$$\bar{t}_B(\theta_B) = E_{\theta_S}\left[t_B(\theta_S,\theta_B)\right] = -E\left[\overline{p}_S(\theta_S)\phi_S^1(\theta_S)\right] + E[\overline{p}_B(\theta_B)\psi_B^1(\theta_B)]$$
$$- \overline{p}_B(\theta_B)\theta_B + \int_{\underline{\theta}_B}^{\theta_B} \overline{p}_B(z)dz.$$

Hence, (IC2) is satisfied with $U_B(\underline{\theta}_B) = E\left[p(\theta_S,\theta_B)\left(\psi_B^1(\theta_B) - \phi_S^1(\theta_S)\right)\right].$
Since (11) holds, this is also interim IR for the buyer. □

Finding the optimal allocation rule

Thanks to the lemma, we can restate the optimization problem as

$$\max_{p} \; E\big[p(\theta_S, \theta_B)(\theta_B - \theta_S)\big]$$

subject to

1. $\overline{p}_S(\theta_S)$ is non-increasing and $\overline{p}_B(\theta_B)$ is non-decreasing,
2. $E\big[p(\theta_S, \theta_B)(\psi_B^1(\theta_B) - \phi_S^1(\theta_S))\big] \geq 0.$

We will ignore the 1st constraint and maximize the objective only wrt the 2nd. Assuming (R2), we will find that the solution also satisfies the 1st constraint.

- The ex post efficient allocation p^* maximizes the objective under the 1st constraint. Hence, by the Myerson-Satterthwaite Theorem, the 2nd constraint has to be violated. $\Rightarrow$ This constraint is binding in the optimum.
- We will solve

  $\max_{p} \; E\big[p(\theta_S, \theta_B)(\theta_B - \theta_S)\big]$ s.t. $E\big[p(\theta_S, \theta_B)(\psi_B^1(\theta_B) - \phi_S^1(\theta_S))\big] = 0.$

- The **Lagrange function** associated with this maximization problem is

$$\mathcal{L}(p,\lambda) = E\big[p(\theta_S,\theta_B)(\theta_B-\theta_S)\big] + \lambda E\big[p(\theta_S,\theta_B)\big(\psi_B^1(\theta_B)-\phi_S^1(\theta_S)\big)\big]$$

$\rightarrow$ We have to find p and $\lambda > 0$ such that $\mathcal{L}$ is maximized and $E\big[p(\theta_S,\theta_B)\big(\psi_B^1(\theta_B)-\phi_S^1(\theta_S)\big)\big] = 0$.

- The Lagrange function can be written as

$$\begin{aligned}\mathcal{L}(p,\lambda) &= E\Big[p(\theta_S,\theta_B)\Big(\theta_B+\lambda\psi_B^1(\theta_B)-\theta_S-\lambda\phi_S^1(\theta_S)\Big)\Big] \\ &= (1+\lambda)E\left[p(\theta_S,\theta_B)\left(\psi_B^{\frac{\lambda}{1+\lambda}}(\theta_B)-\phi_S^{\frac{\lambda}{1+\lambda}}(\theta_S)\right)\right].\end{aligned}$$

- For a given $\lambda \geq 0$, let $\alpha = \frac{\lambda}{1+\lambda}$. $\mathcal{L}$ is maximized by the allocation rule

$$p^\alpha(\theta_S,\theta_B) := \begin{cases} 1 & \text{if } \psi_B^\alpha(\theta_B) > \phi_S^\alpha(\theta_S), \\ 0 & \text{if } \psi_B^\alpha(\theta_B) < \phi_S^\alpha(\theta_S). \end{cases}$$

- Define $\Omega(\alpha) := E\big[p^\alpha(\theta_S, \theta_B)\big(\psi_B^1(\theta_B) - \phi_S^1(\theta_S)\big)\big]$.

  Ω has the following properties:

  - By the Myerson-Satterthwaite Theorem, $\Omega(0) < 0$. Moreover, $\Omega(1) > 0$.
  - $\Omega(\alpha)$ is strictly increasing in α.
    (As α increases, p^α prevents more trades where $\psi_B^1(\theta_B) < \phi_S^1(\theta_S)$.)

- There is a unique $\alpha \in (0,1)$ such that $\Omega(\alpha) = 0$, i.e.,
  a unique $\lambda > 0$ such that $E\big[p^{\frac{\lambda}{1+\lambda}}(\theta_S, \theta_B)\big(\psi_B^1(\theta_B) - \phi_S^1(\theta_S)\big)\big] = 0$.

Suppose regularity assumption (R2) holds. Then, for all $\alpha \in (0,1)$, the allocation rule $p^\alpha(\theta_S, \theta_B)$ satisfies the 1st constraint:
$\overline{p}_S^\alpha(\theta_S)$ is non-increasing in θ_S and $\overline{p}_B^\alpha(\theta_B)$ is non-decreasing in θ_B.

The main result

Proposition (Myerson and Satterthwaite, 1983)

Assume (R2). All mechanisms that maximize the ex ante expected gains from trade under budget balance and interim individual rationality Bayesian implement the following allocation rule:

$$p^\alpha(\theta_S, \theta_B) := \begin{cases} 1 & \textit{if } \phi_S^\alpha(\theta_S) < \psi_B^\alpha(\theta_B), \\ 0 & \textit{if } \phi_S^\alpha(\theta_S) > \psi_B^\alpha(\theta_B), \end{cases}$$

where $\alpha \in (0,1)$ *is the unique solution to*

$$E\Big[p^\alpha(\theta_S, \theta_B)\big(\psi_B^1(\theta_B) - \phi_S^1(\theta_S)\big)\Big] = 0.$$

- There are many budget balanced payment rules t that together with the allocation rule p^α yield an optimal direct mechanism. One possibility is the construction from the proof of the lemma (slide 118).

Discussion

- In an optimal mechanism, the object is traded less frequently than ex post efficiency requires: $p^\alpha(\theta_S, \theta_B) = 1$ if and only if

$$\theta_B > \psi_B^\alpha(\theta_B) \geq \phi_S^\alpha(\theta_S) > \theta_S.$$

- The object is only traded if the difference $\theta_B - \theta_S$ is large enough. $\Rightarrow$ Inefficiency is introduced when the gains from trade are relatively small.

- Inefficiency has to be introduced in order to reduce the agents' information rents such that individually rational transfers that do not incur a deficit become possible. An optimal mechanism introduces the required inefficiency in the least costly way (in terms of loss in gains from trade).

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