Introduction to Mechanism Design

Cédric Wasser

April 2015

1 Introduction

1.1 Overview

- How can individual preferences be aggregated into a collective decision?
- Problem: typically, individual preferences are not publicly observable
  $\rightarrow$ individuals must be relied upon to reveal this information
- How does the information revelation problem constrain the ways in which collective decisions can respond to individual preferences?

1.1 Overview

- How can individual preferences be aggregated into a collective decision?
- Problem: typically, individual preferences are not publicly observable
  $\rightarrow$ individuals must be relied upon to reveal this information
- How does the information revelation problem constrain the ways in which collective decisions can respond to individual preferences?

Mechanism Design vs. Game Theory

- *Game Theory*: What is the outcome of strategic interaction between individuals in a **given game**, i.e., economic environment/institution?
- *Mechanism Design*: How do we **design the game**, i.e., economic environment/institution to obtain a certain outcome?

1.1 Overview

- How can individual preferences be aggregated into a collective decision?
- Problem: typically, individual preferences are not publicly observable
  $\rightarrow$ individuals must be relied upon to reveal this information
- How does the information revelation problem constrain the ways in which collective decisions can respond to individual preferences?

Mechanism Design vs. Game Theory

- *Game Theory*: What is the outcome of strategic interaction between individuals in a **given game**, i.e., economic environment/institution?
- *Mechanism Design*: How do we **design the game**, i.e., economic environment/institution to obtain a certain outcome?

Reading: Mas-Colell, Whinston, Green (1995): Microeconomic Theory, *Chapter 23*

1.2 Example: A Seller's Problem

- One seller, one buyer
- Seller owns a single indivisible object, valuation 0.
- Buyer has valuation v for object.
- Buyer privately knows v.
  Seller knows that v is drawn from distribution F with support $[0, 1]$.

1.2 Example: A Seller's Problem

- One seller, one buyer
- Seller owns a single indivisible object, valuation 0.
- Buyer has valuation v for object.
- Buyer privately knows v.
  Seller knows that v is drawn from distribution F with support $[0, 1]$.

Simple mechanism: **posted price**

- Seller sets price p, buyer decides whether or not to buy at that price.
- Seller's optimal posted price:

$$p^* \in \arg\max_p \, (1 - F(p))p$$

1.2 Example: A Seller's Problem

- One seller, one buyer
- Seller owns a single indivisible object, valuation 0.
- Buyer has valuation v for object.
- Buyer privately knows v.
  Seller knows that v is drawn from distribution F with support $[0, 1]$.

Simple mechanism: **posted price**

- Seller sets price p, buyer decides whether or not to buy at that price.
- Seller's optimal posted price:

$$p^* \in \arg\max_p \, (1 - F(p))p$$

- Revenue maximization ($p = p^* > 0$) vs. efficient allocation ($p = 0$)

Could the Seller do better?

- Seller could use arbitrarily complicated selling procedure, e.g.,
  - negotiation (and renegotiation)
  - offer buyer lotteries at different prices
- Given value v, buyer optimally chooses actions in the selling procedure.
  $\rightarrow$ Any selling procedure results in the buyer obtaining the object with some probability $q(v)$ and paying some amount $t(v)$.

Could the Seller do better?

- Seller could use arbitrarily complicated selling procedure, e.g.,
  - negotiation (and renegotiation)
  - offer buyer lotteries at different prices
- Given value v, buyer optimally chooses actions in the selling procedure.
  $\rightarrow$ Any selling procedure results in the buyer obtaining the object with some probability $q(v)$ and paying some amount $t(v)$.

Revelation principle (informal): For every selling procedure, there is an incentive compatible direct selling mechanism that results in the same outcome.

Could the Seller do better?

- Seller could use arbitrarily complicated selling procedure, e.g.,
  - negotiation (and renegotiation)
  - offer buyer lotteries at different prices
- Given value v, buyer optimally chooses actions in the selling procedure.
  $\rightarrow$ Any selling procedure results in the buyer obtaining the object with some probability $q(v)$ and paying some amount $t(v)$.

Revelation principle (informal): For every selling procedure, there is an incentive compatible direct selling mechanism that results in the same outcome.

Direct selling mechanism (q, t):

1. Buyer reports valuation $\tilde{v}$.
2. Buyer obtains object with probability $q(\tilde{v})$ and pays $t(\tilde{v})$.

(q, t) is **incentive compatible** if it is optimal for the buyer to report his true value v.

Optimal direct selling mechanism

Revelation principle greatly simplifies seller's problem:
Can restrict search for optimal selling procedure to direct selling mechanisms!

At this stage, we will not solve the full problem (we will come back to it later).
$\rightarrow$ Restrict attention to $q(v) \in \{0, 1\}$, i.e., *non-stochastic allocation*.

Optimal direct selling mechanism

Revelation principle greatly simplifies seller's problem:
Can restrict search for optimal selling procedure to direct selling mechanisms!

At this stage, we will not solve the full problem (we will come back to it later).
$\rightarrow$ Restrict attention to $q(v) \in \{0, 1\}$, i.e., *non-stochastic allocation*.

Incentive compatibility (IC):

$$q(v)v - t(v) \geq q(v')v - t(v') \quad \text{for all } v, v'.$$

Optimal direct selling mechanism

Revelation principle greatly simplifies seller's problem:
Can restrict search for optimal selling procedure to direct selling mechanisms!

At this stage, we will not solve the full problem (we will come back to it later).
$\rightarrow$ Restrict attention to $q(v) \in \{0, 1\}$, i.e., *non-stochastic allocation*.

Incentive compatibility (IC):

$$q(v)v - t(v) \geq q(v')v - t(v') \quad \text{for all } v, v'.$$

Implications of IC:

1. $q(v)$ has to be monotone.
   - If $q(v) = 1$ is incentive compatible for v,
     then we must have $q(v') = 1$ for all $v' > v$ as well.

Optimal direct selling mechanism

Revelation principle greatly simplifies seller's problem:
Can restrict search for optimal selling procedure to direct selling mechanisms!

At this stage, we will not solve the full problem (we will come back to it later).
$\rightarrow$ Restrict attention to $q(v) \in \{0, 1\}$, i.e., *non-stochastic allocation*.

Incentive compatibility (IC):

$$q(v)v - t(v) \geq q(v')v - t(v') \quad \text{for all } v, v'.$$

Implications of IC:

1. $q(v)$ has to be monotone.
   - If $q(v) = 1$ is incentive compatible for v, then we must have $q(v') = 1$ for all $v' > v$ as well.
2. $t(v) = t(v')$ for all v, v' where $q(v) = q(v')$.
   - If $t(v) > t(v')$, v would gain from imitating v'.

Any incentive compatible direct selling mechanism must take the following from: For some $\hat{v} \in [0,1]$ and some t_0,

$$q(v) = \begin{cases} 0 & \text{if } v < \hat{v} \\ 1 & \text{if } v \geq \hat{v} \end{cases} \quad \text{and} \quad t(v) = \begin{cases} t_0 & \text{if } v < \hat{v} \\ t_0 + \hat{v} & \text{if } v \geq \hat{v} \end{cases}$$

Any incentive compatible direct selling mechanism must take the following from:
For some $\hat{v} \in [0, 1]$ and some t_0,

$$q(v) = \begin{cases} 0 & \text{if } v < \hat{v} \\ 1 & \text{if } v \geq \hat{v} \end{cases} \quad \text{and} \quad t(v) = \begin{cases} t_0 & \text{if } v < \hat{v} \\ t_0 + \hat{v} & \text{if } v \geq \hat{v} \end{cases}$$

Individual rationality (IR) (seller cannot force buyer to participate):

$$q(v)v - t(v) \geq 0 \quad \text{for all } v.$$

Any incentive compatible direct selling mechanism must take the following from: For some $\hat{v} \in [0, 1]$ and some t_0,

$$q(v) = \begin{cases} 0 & \text{if } v < \hat{v} \\ 1 & \text{if } v \geq \hat{v} \end{cases} \quad \text{and} \quad t(v) = \begin{cases} t_0 & \text{if } v < \hat{v} \\ t_0 + \hat{v} & \text{if } v \geq \hat{v} \end{cases}$$

Individual rationality (IR) (seller cannot force buyer to participate):

$$q(v)v - t(v) \geq 0 \quad \text{for all } v.$$

Optimal non-stochastic direct selling mechanism: Seller solves

$$\max_{\hat{v}, t_0} F(\hat{v})t_0 + (1 - F(\hat{v}))(t_0 + \hat{v}) \quad \text{subject to IR.}$$

IR implies $t_0 = 0$.

Any incentive compatible direct selling mechanism must take the following from: For some $\hat{v} \in [0,1]$ and some t_0,

$$q(v) = \begin{cases} 0 & \text{if } v < \hat{v} \\ 1 & \text{if } v \geq \hat{v} \end{cases} \quad \text{and} \quad t(v) = \begin{cases} t_0 & \text{if } v < \hat{v} \\ t_0 + \hat{v} & \text{if } v \geq \hat{v} \end{cases}$$

Individual rationality (IR) (seller cannot force buyer to participate):

$$q(v)v - t(v) \geq 0 \quad \text{for all } v.$$

Optimal non-stochastic direct selling mechanism: Seller solves

$$\max_{\hat{v}, t_0} F(\hat{v})t_0 + (1 - F(\hat{v}))(t_0 + \hat{v}) \quad \text{subject to IR.}$$

IR implies $t_0 = 0$.

$\Rightarrow$ Optimal allocation and payment is the same as in posted price mechanism!

More Buyers

What if there are $n \geq 2$ potential buyers?

- Seller could use optimal posted price $p^* \in \arg\max_p \ (1 - F(p)^n)p$
- ...or use an **auction** instead ($\rightarrow$ induces **game** between buyers)
- ...or some arbitrarily complicated other selling procedure

More Buyers

What if there are $n \geq 2$ potential buyers?

- Seller could use optimal posted price $p^* \in \arg\max_p \ (1 - F(p)^n)p$
- ...or use an **auction** instead ($\rightarrow$ induces **game** between buyers)
- ...or some arbitrarily complicated other selling procedure

Mechanism design theory will enable us to

- determine revenue maximizing mechanisms for the seller
- determine whether there are mechanisms that allocate the object Pareto efficiently (and characterize such mechanisms)
- identify settings where different auction formats yield same revenue

2 The Mechanism Design Problem

2.1 Environment

- n **agents** $i \in N := \{1, \ldots, n\}$
- set of possible **alternatives** X
- Each agent i has **private information** $\theta_i \in \Theta_i$. (θ_i is agent i's *type*.)
- Each agent i is an expected utility maximizer with vNM utility function

$$u_i(x, \theta) \quad \text{where } x \in X \text{ and } \theta \in \Theta := \Theta_1 \times \cdots \times \Theta_n.$$

2.1 Environment

- n **agents** $i \in N := \{1, \ldots, n\}$
- set of possible **alternatives** X
- Each agent i has **private information** $\theta_i \in \Theta_i$. (θ_i is agent i's *type*.)
- Each agent i is an expected utility maximizer with vNM utility function

$$u_i(x, \theta) \quad \text{where } x \in X \text{ and } \theta \in \Theta := \Theta_1 \times \cdots \times \Theta_n.$$

- Type profile $\theta = (\theta_1, \ldots, \theta_n)$ is drawn from **commonly known distribution** with probability density $f(\cdot)$ over Θ.

2.1 Environment

- n **agents** $i \in N := \{1, \ldots, n\}$
- set of possible **alternatives** X
- Each agent i has **private information** $\theta_i \in \Theta_i$. (θ_i is agent i's *type*.)
- Each agent i is an expected utility maximizer with vNM utility function

$$u_i(x, \theta) \quad \text{where } x \in X \text{ and } \theta \in \Theta := \Theta_1 \times \cdots \times \Theta_n.$$

- Type profile $\theta = (\theta_1, \ldots, \theta_n)$ is drawn from **commonly known distribution** with probability density $f(\cdot)$ over Θ.

Notation:

- $\Theta_{-i} := \Theta_1 \times \cdots \times \Theta_{i-1} \times \Theta_{i+1} \times \cdots \times \Theta_n$.
- For $\theta_i \in \Theta_i$ and $\theta_{-i} = (\theta_1, \ldots, \theta_{i-1}, \theta_{i+1}, \ldots, \theta_n) \in \Theta_{-i}$,

$$(\theta_i, \theta_{-i}) = (\theta_1, \ldots, \theta_{i-1}, \theta_i, \theta_{i+1}, \ldots, \theta_n).$$

Special case: Independent private values

Two often used assumptions:

1. **private values**:

$$u_i(x, \theta) = u_i(x, \theta_i) \quad \text{for all } i \in N \text{ and all } x \in X.$$

Special case: Independent private values

Two often used assumptions:

1. **private values**:

$$u_i(x, \theta) = u_i(x, \theta_i) \quad \text{for all } i \in N \text{ and all } x \in X.$$

2. **independent types**: types are independently distributed, i.e., there are densities $f_i(\theta_i)$ such that

$$f(\theta) = \prod_{i \in N} f_i(\theta_i) \quad \text{for all } \theta \in \Theta.$$

Special case: Independent private values

Two often used assumptions:

1. **private values**:

$$u_i(x,\theta) = u_i(x,\theta_i) \quad \text{for all } i \in N \text{ and all } x \in X.$$

2. **independent types**: types are independently distributed, i.e., there are densities $f_i(\theta_i)$ such that

$$f(\theta) = \prod_{i \in N} f_i(\theta_i) \quad \text{for all } \theta \in \Theta.$$

In **independent private values environments** both of these assumptions hold.

Example 1: Public project with private values

E.g., building a bridge

- Set of alternatives $X = \{0, 1\} \times \mathbb{R}^n$
- $x = (k, t_1, \ldots, t_n) \in X$:
  - if $k = 0$, bridge is not built; if $k = 1$, bridge is built
  - each agent i obtains monetary transfer t_i

Example 1: Public project with private values

E.g., building a bridge

- Set of alternatives $X = \{0, 1\} \times \mathbb{R}^n$
- $x = (k, t_1, \ldots, t_n) \in X$:
    - if $k = 0$, bridge is not built; if $k = 1$, bridge is built
    - each agent i obtains monetary transfer t_i
- private information: $\theta_i \in \mathbb{R}$ is i's willingness to pay for the bridge.
- utility functions:

$$u_i(x, \theta) = u_i(x, \theta_i) = \theta_i k + t_i$$

Example 2a: Auction without externalities

- Auction for one object, two bidders: $N = \{1, 2\}$, $X = \{0, 1, 2\} \times \mathbb{R}^2$
- $(k, t_1, t_2) \in X$:
  - if $k = 0$, object is not sold; if $k = i$, buyer i gets object
  - $-t_i$ is payment by buyer i

Example 2a: Auction without externalities

- Auction for one object, two bidders: $N = \{1, 2\}$, $X = \{0, 1, 2\} \times \mathbb{R}^2$
- $(k, t_1, t_2) \in X$:
  - if $k = 0$, object is not sold; if $k = i$, buyer i gets object
  - $-t_i$ is payment by buyer i
- private information: valuation for the object $\theta_i \in [0, 1]$.
- utility functions:

$$u_i((k, t_1, t_2), \theta) = \begin{cases} \theta_i + t_i & \text{if } k = i \\ t_i & \text{if } k \neq i \end{cases}$$

Example 2b: Auction with allocation externalities

- Environment as in example 2a, but with different types and utilities.
- private information: $\theta_i = (\theta_i^i, \theta_i^j) \in [0,1] \times [-1,0]$
- utility functions:

$$u_i((k, t_1, t_2), \theta) = \begin{cases} \theta_i^i + t_i & \text{if } k = i \\ \theta_i^j + t_i & \text{if } k = j \neq i,\, j \neq 0 \\ t_i & \text{if } k = 0 \end{cases}$$

Example 2b: Auction with allocation externalities

- Environment as in example 2a, but with different types and utilities.
- private information: $\theta_i = (\theta_i^i, \theta_i^j) \in [0,1] \times [-1,0]$
- utility functions:

$$u_i((k, t_1, t_2), \theta) = \begin{cases} \theta_i^i + t_i & \text{if } k = i \\ \theta_i^j + t_i & \text{if } k = j \neq i,\ j \neq 0 \\ t_i & \text{if } k = 0 \end{cases}$$

Example: object is a patent for new product in an oligopolistic market. If competitor of firm i obtains the patent, i's profits are lower than if nobody obtains the patent.

Example 3: Bilateral trade with interdependent values

- $N = \{1, 2\}$ where agent 1 is the owner of an object; $X = \{1, 2\} \times \mathbb{R}^2$.
- $x = (k, t_1, t_2) \in X$: if $k = 1$, agent 1 keeps object, if $k = 2$ object is given to agent 2; t_i is a monetary transfer to agent i.

Example 3: Bilateral trade with interdependent values

- $N = \{1, 2\}$ where agent 1 is the owner of an object; $X = \{1, 2\} \times \mathbb{R}^2$.
- $x = (k, t_1, t_2) \in X$: if $k = 1$, agent 1 keeps object, if $k = 2$ object is given to agent 2; t_i is a monetary transfer to agent i.
- Private information:
    - $\theta_1 = (q, v_1) \in [0, 1] \times [0, 1]$, where q is the quality of the object and v_1 is the owners taste for quality.
    - $\theta_2 = v_2 \in [0, 1]$, where v_2 is the buyer's taste for quality.

Example 3: Bilateral trade with interdependent values

- $N = \{1, 2\}$ where agent 1 is the owner of an object; $X = \{1, 2\} \times \mathbb{R}^2$.
- $x = (k, t_1, t_2) \in X$: if $k = 1$, agent 1 keeps object, if $k = 2$ object is given to agent 2; t_i is a monetary transfer to agent i.
- Private information:
  - $\theta_1 = (q, v_1) \in [0, 1] \times [0, 1]$, where q is the quality of the object and v_1 is the owners taste for quality.
  - $\theta_2 = v_2 \in [0, 1]$, where v_2 is the buyer's taste for quality.
- utility functions:

$$u_1(x, \theta) = \begin{cases} qv_1 + t_1 & \text{if } k = 1 \\ t_1 & \text{if } k = 2 \end{cases} \qquad u_2(x, \theta) = \begin{cases} t_2 & \text{if } k = 1 \\ qv_2 + t_2 & \text{if } k = 2 \end{cases}$$

(buyer's utility depends on seller's private information)

2.2 Social Choice Functions and Mechanisms

Definition

A **social choice function** (SCF) is a function $c\colon \Theta \to X$
that, for each possible type profile θ, chooses an alternative $c(\theta) \in X$.

2.2 Social Choice Functions and Mechanisms

Definition

A **social choice function** (SCF) is a function $c\colon \Theta \to X$ that, for each possible type profile θ, chooses an alternative $c(\theta) \in X$.

A desirable property of SCFs is ex post efficiency:

Definition

A SCF c is **ex post efficient** (or Paretian) if there exists no $\theta \in \Theta$ such that for some $x \in X$

$$u_i(x,\theta) \geq u_i(c(\theta),\theta)\ \forall i \quad \text{and} \quad u_i(x,\theta) > u_i(c(\theta),\theta) \text{ for one } i.$$

Mechanisms

Collective choices are usually made indirectly through institutions in which agents interact. A mechanism is the formal representation such an institution.

Definition

A **mechanism** $\Gamma = (S_1, \ldots, S_n, g)$ consists of

- a **strategy set** S_i for each agent $i \in N$
- and an **outcome function** $g\colon S_1 \times \cdots \times S_n \to X$.

Mechanisms

Collective choices are usually made indirectly through institutions in which agents interact. A mechanism is the formal representation such an institution.

Definition

A **mechanism** $\Gamma = (S_1, \dots, S_n, g)$ consists of

- a **strategy set** S_i for each agent $i \in N$
- and an **outcome function** $g\colon S_1 \times \dots \times S_n \to X$.

A mechanism defines the rules of a procedure for making a collective decision:

- S_i: allowed actions of each agent i
  (e.g., the bids in an auction; the allowable votes in an election)
- g: rule for how agents' actions are turned into a social choice
  (e.g., allocation and payments as a function of bids; set of elected candidates)

Mechanisms

Collective choices are usually made indirectly through institutions in which agents interact. A mechanism is the formal representation such an institution.

Definition

A **mechanism** $\Gamma = (S_1, \ldots, S_n, g)$ consists of

- a **strategy set** S_i for each agent $i \in N$
- and an **outcome function** $g\colon S_1 \times \cdots \times S_n \to X$.

A mechanism defines the rules of a procedure for making a collective decision:

- S_i: allowed actions of each agent i
  (e.g., the bids in an auction; the allowable votes in an election)
- g: rule for how agents' actions are turned into a social choice
  (e.g., allocation and payments as a function of bids; set of elected candidates)

A mechanism need not be static.
(e.g., an auction/election may involve several rounds of bidding/voting)

The induced game of incomplete information

A mechanism Γ combined with the environment **induces** a **Bayesian game** $G_\Gamma := [N, \{S_i\}_{i \in N}, \{\tilde{u}_i\}_{i \in N}, \Theta, f(\cdot)]$ with payoffs

$$\tilde{u}_i(s_1, \ldots, s_n, \theta) := u_i(g(s_1, \ldots, s_n), \theta) \quad \forall (s_1, \ldots, s_n) \in S_1 \times \cdots \times S_n.$$

The induced game of incomplete information

A mechanism Γ combined with the environment **induces** a **Bayesian game** $G_\Gamma := [N, \{S_i\}_{i\in N}, \{\tilde{u}_i\}_{i\in N}, \Theta, f(\cdot)]$ with payoffs

$$\tilde{u}_i(s_1, \ldots, s_n, \theta) := u_i(g(s_1, \ldots, s_n), \theta) \quad \forall (s_1, \ldots, s_n) \in S_1 \times \cdots \times S_n.$$

A strategy $s_i \colon \Theta_i \to S_i$ for agent i specifies a choice $s_i(\theta_i)$ for each type θ_i.

We will use two equilibrium concepts: A strategy profile $(s_1^*(\cdot), \ldots, s_n^*(\cdot))$ is

- a **dominant strategy equilibrium** if, for each $i \in N$ and $\theta \in \Theta$,

$$\tilde{u}_i(s_i^*(\theta_i), s_{-i}, \theta) \geq \tilde{u}_i(s_i', s_{-i}, \theta) \quad \forall s_i' \in S_i \text{ and } s_{-i} \in S_{-i}.$$

The induced game of incomplete information

A mechanism Γ combined with the environment **induces** a **Bayesian game** $G_\Gamma := [N, \{S_i\}_{i\in N}, \{\tilde{u}_i\}_{i\in N}, \Theta, f(\cdot)]$ with payoffs

$$\tilde{u}_i(s_1,\ldots,s_n,\theta) := u_i(g(s_1,\ldots,s_n),\theta) \quad \forall (s_1,\ldots,s_n) \in S_1 \times \cdots \times S_n.$$

A strategy $s_i\colon \Theta_i \to S_i$ for agent i specifies a choice $s_i(\theta_i)$ for each type θ_i.

We will use two equilibrium concepts: A strategy profile $(s_1^*(\cdot),\ldots,s_n^*(\cdot))$ is

- a **dominant strategy equilibrium** if, for each $i \in N$ and $\theta \in \Theta$,

$$\tilde{u}_i(s_i^*(\theta_i), s_{-i}, \theta) \geq \tilde{u}_i(s_i', s_{-i}, \theta) \quad \forall s_i' \in S_i \text{ and } s_{-i} \in S_{-i}.$$

- a **Bayesian Nash equilibrium** if, for each $i \in N$ and $\theta_i \in \Theta_i$,

$$E_{\theta_{-i}}\big[\tilde{u}_i(s_i^*(\theta_i), s_{-i}^*(\theta_{-i}), \theta)\big|\theta_i\big] \geq E_{\theta_{-i}}\big[\tilde{u}_i(\hat{s}_i, s_{-i}^*(\theta_{-i}), \theta)\big|\theta_i\big] \quad \forall \hat{s}_i \in S_i.$$

Implementation

Definition

The mechanism $\Gamma = (S_1, \ldots, S_n, g)$ **implements** the SCF c if there is an equilibrium strategy profile $(s_1^*(\cdot), \ldots, s_n^*(\cdot))$ of the induced game G_Γ such that

$$g(s_1^*(\theta_1), \ldots, s_n^*(\theta_n)) = c(\theta) \quad \text{for all } \theta \in \Theta.$$

If Γ implements c, we say c is a **performance** of Γ.

Implementation

Definition

The mechanism $\Gamma = (S_1, \dots, S_n, g)$ **implements** the SCF c if there is an equilibrium strategy profile $(s_1^*(\cdot), \dots, s_n^*(\cdot))$ of the induced game G_Γ such that

$$g(s_1^*(\theta_1), \dots, s_n^*(\theta_n)) = c(\theta) \quad \text{for all } \theta \in \Theta.$$

If Γ implements c, we say c is a **performance** of Γ.

- G_Γ may have several equilibria with different performances.
  We only require that the outcome of one equilibrium coincides with c.
- Depending on the equilibrium concept we use, we say
  - either: Γ implements c in dominant strategies; c is a dominant strategy performance of Γ.
  - or: Γ implements c in Bayesian Nash equilibrium; c is a Bayesian performance of Γ.

Example: Second-price auction

Single object is auctioned among n bidders using a second-price auction: the highest bidder wins, paying the second-highest bid.

Environment:

- Bidders $N = \{1, \ldots, n\}$, alternatives $X = \{0, 1, \ldots, n\} \times \mathbb{R}^n$
- $(k, t_1, \ldots, t_n) \in X$:
  - if $k = 0$, object is not sold; if $k = i$, bidder i gets object
  - $-t_i$ is payment by bidder i.
- types: $\Theta_i = [0, 1]$ for all i
- utility functions:

$$u_i((k, t_1, \ldots, t_n), \theta_i) = \begin{cases} \theta_i + t_i & \text{if } k = i \\ t_i & \text{if } k \neq i \end{cases}$$

Mechanism $\Gamma = (S_1, \ldots, S_n, g)$:

- For each i, the strategy set $S_i = \mathbb{R}_+$ is the set of possible bids.
- For each profile of bids $s = (s_1, \ldots, s_n) \in \mathbb{R}_+^n$,
  the outcome is $g(s) = (k(s), t_1(s), \ldots, t_n(s))$ where

$$k(s) = \min\{i \in N \mid s_i \geq s_j \,\forall j \in N\}$$

and

$$t_i(s) = \begin{cases} -\max_{j \neq i} s_j & \text{if } k(s) = i \\ 0 & \text{otherwise} \end{cases}$$

Mechanism $\Gamma = (S_1, \ldots, S_n, g)$:

- For each i, the strategy set $S_i = \mathbb{R}_+$ is the set of possible bids.
- For each profile of bids $s = (s_1, \ldots, s_n) \in \mathbb{R}_+^n$,
  the outcome is $g(s) = (k(s), t_1(s), \ldots, t_n(s))$ where

$$k(s) = \min\{i \in N \mid s_i \geq s_j \; \forall j \in N\}$$

and

$$t_i(s) = \begin{cases} -\max_{j \neq i} s_j & \text{if } k(s) = i \\ 0 & \text{otherwise} \end{cases}$$

Dominant strategy equilibrium: $s_i^*(\theta_i) = \theta_i$ for all i.

Mechanism $\Gamma = (S_1, \ldots, S_n, g)$:

- For each i, the strategy set $S_i = \mathbb{R}_+$ is the set of possible bids.
- For each profile of bids $s = (s_1, \ldots, s_n) \in \mathbb{R}^n_+$, the outcome is $g(s) = (k(s), t_1(s), \ldots, t_n(s))$ where

$$k(s) = \min\{i \in N \mid s_i \geq s_j \; \forall j \in N\}$$

and

$$t_i(s) = \begin{cases} -\max_{j \neq i} s_j & \text{if } k(s) = i \\ 0 & \text{otherwise} \end{cases}$$

Dominant strategy equilibrium: $s_i^*(\theta_i) = \theta_i$ for all i.

$c(\theta_1, \ldots, \theta_n) = g(s_1^*(\theta_1), \ldots, s_n^*(\theta_n)) = g(\theta_1, \ldots, \theta_n)$ is a dominant strategy performance of the second-price auction.

2.3 Direct Mechanisms and the Revelation Principle

Central Question: Which social choice functions are implementable in an environment with private information?

2.3 Direct Mechanisms and the Revelation Principle

Central Question: Which social choice functions are implementable in an environment with private information?

A SCF c depends on private information that agents cannot be forced to reveal.

But it may be possible to *design* a mechanism that implements the SCF c.
$\rightarrow$ A SCF is **implementable** if there exists a mechanism that implements it.

2.3 Direct Mechanisms and the Revelation Principle

Central Question: Which social choice functions are implementable in an environment with private information?

A SCF c depends on private information that agents cannot be forced to reveal.

But it may be possible to *design* a mechanism that implements the SCF c.
$\rightarrow$ A SCF is **implementable** if there exists a mechanism that implements it.

- Are ex post efficient SCFs implementable?
- Which implementable SCF maximizes a given objective?
  (e.g. expected welfare or utility of mechanism designer)

Commitment

We assume that the mechanism designer has full commitment power: he can set the rules of the mechanism and commit that he will not change the rules after the agents have chosen their actions.

Commitment

We assume that the mechanism designer has full commitment power: he can set the rules of the mechanism and commit that he will not change the rules after the agents have chosen their actions.

Example: the mechanism designer is the seller in an auction

- The seller commits to refuse any renegotiation after the auction, e.g., if a non-winning bidder offers to pay more than the winner has to.
- In a second-price auction, the seller has to credibly commit to only charge the second-highest bid form the winner.

Commitment

We assume that the mechanism designer has full commitment power: he can set the rules of the mechanism and commit that he will not change the rules after the agents have chosen their actions.

Example: the mechanism designer is the seller in an auction

- The seller commits to refuse any renegotiation after the auction, e.g., if a non-winning bidder offers to pay more than the winner has to.
- In a second-price auction, the seller has to credibly commit to only charge the second-highest bid form the winner.

Sources of commitment:

- Contracts
- Reputation / repeated play

Direct Mechanisms

Problem: Set of possible mechanisms is extremely large.

Direct Mechanisms

Problem: Set of possible mechanisms is extremely large.

A smaller class of mechanisms:

Definition

A mechanism $\Gamma = (S_1, \ldots, S_n, g)$ is a **direct mechanism** if $S_i = \Theta_i$ for all i.

Direct Mechanisms

Problem: Set of possible mechanisms is extremely large.

A smaller class of mechanisms:

Definition

A mechanism $\Gamma = (S_1, \dots, S_n, g)$ is a **direct mechanism** if $S_i = \Theta_i$ for all i.

Definition

A direct mechanism Γ is dominant strategy (Bayesian) **incentive compatible** if $(s_1^*(\cdot), \dots, s_n^*(\cdot))$ with $s_i^*(\theta_i) = \theta_i$ for all $\theta_i \in \Theta_i$ and $i \in N$ is a dominant strategy (Bayesian Nash) equilibrium of the game G_Γ induced by Γ.

Truthful implementation

Definition

A SCF is **truthfully implementable** in dominant strategies (in Bayesian Nash equilibrium) if it is the performance of a dominant strategy (Bayesian) incentive compatible direct mechanism.

Truthful implementation

Definition

A SCF is **truthfully implementable** in dominant strategies (in Bayesian Nash equilibrium) if it is the performance of a dominant strategy (Bayesian) incentive compatible direct mechanism.

Remarks:

- Equivalent definition: c is truthfully implementable if $\Gamma = (\Theta_1, \ldots, \Theta_n, c)$ is an incentive compatible direct mechanism.
- A SCF that is truthfully implementable in dominant strategies is also called *strategy-proof*.

The Revelation Principle

Proposition

Let $\Gamma = (S_1, \ldots, S_n, g)$ *be any mechanism with dominant strategy (Bayesian) performance* c_Γ*. Then* $\Gamma' = (\Theta_1, \ldots, \Theta_n, c_\Gamma)$ *is a dominant strategy (Bayesian) incentive compatible direct mechanism.*

The Revelation Principle

Proposition

Let $\Gamma = (S_1, \ldots, S_n, g)$ *be any mechanism with dominant strategy (Bayesian) performance* c_Γ. *Then* $\Gamma' = (\Theta_1, \ldots, \Theta_n, c_\Gamma)$ *is a dominant strategy (Bayesian) incentive compatible direct mechanism.*

Corollary (Revelation Principle)

A SCF is implementable if and only if it is truthfully implementable.

The Revelation Principle

Proposition

Let $\Gamma = (S_1, \ldots, S_n, g)$ *be any mechanism with dominant strategy (Bayesian) performance* c_Γ*. Then* $\Gamma' = (\Theta_1, \ldots, \Theta_n, c_\Gamma)$ *is a dominant strategy (Bayesian) incentive compatible direct mechanism.*

Corollary (Revelation Principle)

A SCF is implementable if and only if it is truthfully implementable.

(For private values environments with unrestricted preferences and $|X| > 2$, the *Gibbard-Satterthwaite Theorem* combined with the *Revelation Principle* implies that only *dictatorial* SCFs are implementable in dominant strategies.

$\rightarrow$ In the next section, we will restrict preferences by assuming quasi-linearity.)

We will only prove the dominant strategy version of the proposition.
(The Bayesian version is left as an exercise.)

Proof.

Since c_Γ is a dominant strategy performance of Γ, there exist dominant strategies $s_1^*(\cdot), \ldots, s_n^*(\cdot)$ in G_Γ such that

$$c_\Gamma(\theta) = g(s_1^*(\theta_1), \ldots, s_n^*(\theta_n)).$$

We will only prove the dominant strategy version of the proposition.
(The Bayesian version is left as an exercise.)

Proof.

Since c_Γ is a dominant strategy performance of Γ, there exist dominant strategies $s_1^*(\cdot), \ldots, s_n^*(\cdot)$ in G_Γ such that

$$c_\Gamma(\theta) = g(s_1^*(\theta_1), \ldots, s_n^*(\theta_n)).$$

Now consider Γ'. Fix agent i and the reports of the other agents $\hat{\theta}_{-i} \in \Theta_{-i}$.
Payoff of agent i with type θ_i and report $\hat{\theta}_i$:

$$\begin{aligned} u_i(c_\Gamma(\hat{\theta}_i, \hat{\theta}_{-i}), \theta) &= u_i(g(s_i^*(\hat{\theta}_i), s_{-i}^*(\hat{\theta}_{-i})), \theta) \\ &\leq u_i(g(s_i^*(\theta_i), s_{-i}^*(\hat{\theta}_{-i})), \theta) = u_i(c_\Gamma(\theta_i, \hat{\theta}_{-i}), \theta) \end{aligned}$$

We will only prove the dominant strategy version of the proposition.
(The Bayesian version is left as an exercise.)

Proof.

Since c_Γ is a dominant strategy performance of Γ, there exist dominant strategies $s_1^*(\cdot), \ldots, s_n^*(\cdot)$ in G_Γ such that

$$c_\Gamma(\theta) = g(s_1^*(\theta_1), \ldots, s_n^*(\theta_n)).$$

Now consider Γ'. Fix agent i and the reports of the other agents $\hat{\theta}_{-i} \in \Theta_{-i}$.
Payoff of agent i with type θ_i and report $\hat{\theta}_i$:

$$\begin{aligned} u_i(c_\Gamma(\hat{\theta}_i, \hat{\theta}_{-i}), \theta) &= u_i(g(s_i^*(\hat{\theta}_i), s_{-i}^*(\hat{\theta}_{-i})), \theta) \\ &\leq u_i(g(s_i^*(\theta_i), s_{-i}^*(\hat{\theta}_{-i})), \theta) = u_i(c_\Gamma(\theta_i, \hat{\theta}_{-i}), \theta) \end{aligned}$$

because $s_i^*(\theta_i)$ is a dominant strategy of type θ_i. Hence, $\hat{\theta}_i = \theta_i$ is a dominant strategy for player i in $G_{\Gamma'}$ □

3 Quasi-Linear Private Values Environments

3.1 Setup

Throughout this section, we assume **quasi-linear** utilities and **private values**.

Each **alternative** $x = (k, t_1, \ldots, t_n) \in X$ consists of

1. a physical **allocation** (or "project choice") $k \in K$,
2. a monetary **transfer** $t_i \in \mathbb{R}$ to each agent i.

3.1 Setup

Throughout this section, we assume **quasi-linear** utilities and **private values**.

Each **alternative** $x = (k, t_1, \dots, t_n) \in X$ consists of

1. a physical **allocation** (or "project choice") $k \in K$,
2. a monetary **transfer** $t_i \in \mathbb{R}$ to each agent i.

Utility function of agent i:

$$u_i((k, t_1, \dots, t_n), \theta) = v_i(k, \theta_i) + t_i.$$

3.1 Setup

Throughout this section, we assume **quasi-linear** utilities and **private values**.

Each **alternative** $x = (k, t_1, \ldots, t_n) \in X$ consists of

1. a physical **allocation** (or "project choice") $k \in K$,
2. a monetary **transfer** $t_i \in \mathbb{R}$ to each agent i.

Utility function of agent i:

$$u_i((k, t_1, \ldots, t_n), \theta) = v_i(k, \theta_i) + t_i.$$

Remarks:

- $v_i(k, \theta_i)$ is the **value** of allocation k to agent i in terms of money.
- Agents are risk-neutral with respect to money (independent of wealth).
- Utility is freely *transferable* across agents.

Feasibility and Social Choice Functions

We assume that there is no outside source of financing (no budget deficit).

$\Rightarrow$ Transfers $t := (t_1, \ldots, t_n)$ are *feasible* if and only if $\sum_{i \in N} t_i \leq 0$.

Feasibility and Social Choice Functions

We assume that there is no outside source of financing (no budget deficit).

$\Rightarrow$ Transfers $t := (t_1, \dots, t_n)$ are *feasible* if and only if $\sum_{i \in N} t_i \leq 0$.

Set of alternatives:

$$X = \left\{ (k, t) \in K \times \mathbb{R}^n \;\middle|\; \sum_{i \in N} t_i \leq 0 \right\}$$

Feasibility and Social Choice Functions

We assume that there is no outside source of financing (no budget deficit).
$\Rightarrow$ Transfers $t := (t_1, \ldots, t_n)$ are *feasible* if and only if $\sum_{i \in N} t_i \leq 0$.

Set of alternatives:

$$X = \left\{ (k, t) \in K \times \mathbb{R}^n \;\middle|\; \sum_{i \in N} t_i \leq 0 \right\}$$

A social choice function (SCF) $c = (k, t)$ consists of

an **allocation rule** $k \colon \Theta \to K$ and a **payment rule** $t \colon \Theta \to \mathbb{R}^n$

that assign an alternative $(k(\theta), t(\theta)) \in X$ to each type profile θ,
where $t(\theta) := (t_1(\theta), \ldots, t_n(\theta))$.

Ex post efficient SCFs

In a quasi-linear environment, a SCF $c = (k, t)$ is ex post efficient if and only if

- allocation rule k is **value maximizing**:

$$\sum_{i \in N} v_i(k(\theta), \theta_i) \geq \sum_{i \in N} v_i(\hat{k}, \theta_i) \quad \text{for all } \hat{k} \in K \text{ and } \theta \in \Theta,$$

Ex post efficient SCFs

In a quasi-linear environment, a SCF $c = (k, t)$ is ex post efficient if and only if

- allocation rule k is **value maximizing**:

$$\sum_{i \in N} v_i(k(\theta), \theta_i) \geq \sum_{i \in N} v_i(\hat{k}, \theta_i) \quad \text{for all } \hat{k} \in K \text{ and } \theta \in \Theta,$$

- and payment rule t satisfies **budget balance**:

$$\sum_{i \in N} t_i(\theta) = 0 \quad \text{for all } \theta \in \Theta.$$

Ex post efficient SCFs

In a quasi-linear environment, a SCF $c = (k, t)$ is ex post efficient if and only if

- allocation rule k is **value maximizing**:

$$\sum_{i \in N} v_i(k(\theta), \theta_i) \geq \sum_{i \in N} v_i(\hat{k}, \theta_i) \quad \text{for all } \hat{k} \in K \text{ and } \theta \in \Theta,$$

- and payment rule t satisfies **budget balance**:

$$\sum_{i \in N} t_i(\theta) = 0 \quad \text{for all } \theta \in \Theta.$$

We next focus on dominant strategy implementation and study value maximization without and with budget balance.

3.2 Value Maximization

Proposition

Let k^* be a value maximizing allocation rule. The SCF $c = (k^*, t)$ is truthfully implementable in dominant strategies if, for all $i \in N$,

$$t_i(\theta) = \sum_{j \neq i} v_j(k^*(\theta), \theta_j) + h_i(\theta_{-i}), \tag{1}$$

where h_i is an arbitrary function $h_i \colon \Theta_{-i} \to \mathbb{R}$.

3.2 Value Maximization

Proposition

Let k^* be a value maximizing allocation rule. The SCF $c = (k^*, t)$ is truthfully implementable in dominant strategies if, for all $i \in N$,

$$t_i(\theta) = \sum_{j \neq i} v_j(k^*(\theta), \theta_j) + h_i(\theta_{-i}), \tag{1}$$

where h_i is an arbitrary function $h_i \colon \Theta_{-i} \to \mathbb{R}$.

Definition

A direct mechanism $\Gamma = (\Theta_1, \ldots, \Theta_n, c)$ with $c = (k^*, t)$ where k^* is value maximizing and t satisfies (1) is a **Vickrey-Clarke-Groves** (VCG) mechanism.

VCG mechanisms are named after Vickrey (1961), Clarke (1971), and Groves (1973).

Proof.

$c = (k^*, t)$ is truthfully implementable in dominant strategies if, for all $i \in N$, all $\theta_i, \hat{\theta}_i \in \Theta_i$, and all $\theta_{-i} \in \Theta_{-i}$,

$$v_i(k^*(\theta_i, \theta_{-i}), \theta_i) + t_i(\theta_i, \theta_{-i}) \geq v_i(k^*(\hat{\theta}_i, \theta_{-i}), \theta_i) + t_i(\hat{\theta}_i, \theta_{-i})$$

Proof.

$c = (k^*, t)$ is truthfully implementable in dominant strategies if, for all $i \in N$, all $\theta_i, \hat{\theta}_i \in \Theta_i$, and all $\theta_{-i} \in \Theta_{-i}$,

$$v_i(k^*(\theta_i, \theta_{-i}), \theta_i) + t_i(\theta_i, \theta_{-i}) \geq v_i(k^*(\hat{\theta}_i, \theta_{-i}), \theta_i) + t_i(\hat{\theta}_i, \theta_{-i})$$

$$\iff \sum_{j \in N} v_j(k^*(\theta_i, \theta_{-i}), \theta_j) \geq \sum_{j \in N} v_j(k^*(\hat{\theta}_i, \theta_{-i}), \theta_j).$$

Proof.

$c = (k^*, t)$ is truthfully implementable in dominant strategies if, for all $i \in N$, all $\theta_i, \hat{\theta}_i \in \Theta_i$, and all $\theta_{-i} \in \Theta_{-i}$,

$$v_i(k^*(\theta_i, \theta_{-i}), \theta_i) + t_i(\theta_i, \theta_{-i}) \geq v_i(k^*(\hat{\theta}_i, \theta_{-i}), \theta_i) + t_i(\hat{\theta}_i, \theta_{-i})$$

$$\iff \sum_{j \in N} v_j(k^*(\theta_i, \theta_{-i}), \theta_j) \geq \sum_{j \in N} v_j(k^*(\hat{\theta}_i, \theta_{-i}), \theta_j).$$

This is fulfilled because $k^*(\theta_i, \theta_{-i})$ maximizes $\sum_{j \in N} v_j(k, \theta_j)$ for all $\theta \in \Theta$. □

Proof.

$c = (k^*, t)$ is truthfully implementable in dominant strategies if, for all $i \in N$, all $\theta_i, \hat{\theta}_i \in \Theta_i$, and all $\theta_{-i} \in \Theta_{-i}$,

$$v_i(k^*(\theta_i, \theta_{-i}), \theta_i) + t_i(\theta_i, \theta_{-i}) \geq v_i(k^*(\hat{\theta}_i, \theta_{-i}), \theta_i) + t_i(\hat{\theta}_i, \theta_{-i})$$

$$\iff \sum_{j \in N} v_j(k^*(\theta_i, \theta_{-i}), \theta_j) \geq \sum_{j \in N} v_j(k^*(\hat{\theta}_i, \theta_{-i}), \theta_j).$$

This is fulfilled because $k^*(\theta_i, \theta_{-i})$ maximizes $\sum_{j \in N} v_j(k, \theta_j)$ for all $\theta \in \Theta$. □

Intuition: The transfer to an agent i who reports $\hat{\theta}_i$ consists of two parts.

1. $\sum_{j \neq i} v_j(k^*(\hat{\theta}_i, \theta_{-i}), \theta_j)$ is used to equate i's payoff with the total value. Hence, i's incentives are aligned with the goal of value maximization.
2. $h_i(\theta_{-i})$ does not distort incentives because it is independent of i's report.

The Pivot Mechanism

Let $k^*_{-i}(\theta_{-i})$ be an allocation rule that maximizes the value of all agents $j \neq i$:

$$\sum_{j \neq i} v_j(k^*_{-i}(\theta_{-i}), \theta_j) \geq \sum_{j \neq i} v_j(k, \theta_j) \quad \text{for all } k \in K \text{ and } \theta_{-i} \in \Theta_{-i}.$$

The Pivot Mechanism

Let $k^*_{-i}(\theta_{-i})$ be an allocation rule that maximizes the value of all agents $j \neq i$:

$$\sum_{j \neq i} v_j(k^*_{-i}(\theta_{-i}), \theta_j) \geq \sum_{j \neq i} v_j(k, \theta_j) \quad \text{for all } k \in K \text{ and } \theta_{-i} \in \Theta_{-i}.$$

Definition (Clarke, 1971)

The **pivot mechanism** is a VCG mechanism with

$$h_i(\theta_{-i}) = -\sum_{j \neq i} v_j(k^*_{-i}(\theta_{-i}), \theta_j).$$

The Pivot Mechanism

Let $k^*_{-i}(\theta_{-i})$ be an allocation rule that maximizes the value of all agents $j \neq i$:

$$\sum_{j \neq i} v_j(k^*_{-i}(\theta_{-i}), \theta_j) \geq \sum_{j \neq i} v_j(k, \theta_j) \quad \text{for all } k \in K \text{ and } \theta_{-i} \in \Theta_{-i}.$$

Definition (Clarke, 1971)

The **pivot mechanism** is a VCG mechanism with

$$h_i(\theta_{-i}) = -\sum_{j \neq i} v_j(k^*_{-i}(\theta_{-i}), \theta_j).$$

- In the pivot mechanism, the transfer to agent i is defined to be equal to the *externality* i imposes on the other agents:

$$t_i(\theta) = \sum_{j \neq i} \Big(v_j(k^*(\theta), \theta_j) - v_j(k^*_{-i}(\theta_{-i}), \theta_j) \Big)$$

Depending on the functions $h_i(\cdot)$, the payment rule t of a VCG mechanism may not be feasible and violate $\sum_{i \in N} t_i(\theta) \leq 0$. Example:

- $h_i(\theta_{-i}) = 0 \; \forall i$ leads to a budget deficit of $(n-1) \sum_{i \in N} v_i(k^*(\theta), \theta_i)$.

Depending on the functions $h_i(\cdot)$, the payment rule t of a VCG mechanism may not be feasible and violate $\sum_{i \in N} t_i(\theta) \leq 0$. Example:

- $h_i(\theta_{-i}) = 0 \; \forall i$ leads to a budget deficit of $(n-1) \sum_{i \in N} v_i(k^*(\theta), \theta_i)$.

Proposition

The payment rule of the pivot mechanism satisfies $\sum_{i \in N} t_i(\theta) \leq 0$ for all $\theta \in \Theta$.

Depending on the functions $h_i(\cdot)$, the payment rule t of a VCG mechanism may not be feasible and violate $\sum_{i \in N} t_i(\theta) \leq 0$. Example:

- $h_i(\theta_{-i}) = 0 \; \forall i$ leads to a budget deficit of $(n-1) \sum_{i \in N} v_i(k^*(\theta), \theta_i)$.

Proposition

The payment rule of the pivot mechanism satisfies $\sum_{i \in N} t_i(\theta) \leq 0$ for all $\theta \in \Theta$.

Proof.

$$\sum_{i \in N} t_i(\theta) = \sum_{i \in N} \left(\sum_{j \neq i} v_j(k^*(\theta), \theta_j) - \sum_{j \neq i} v_j(k^*_{-i}(\theta_{-i}), \theta_j) \right)$$

Depending on the functions $h_i(\cdot)$, the payment rule t of a VCG mechanism may not be feasible and violate $\sum_{i \in N} t_i(\theta) \leq 0$. Example:

- $h_i(\theta_{-i}) = 0 \; \forall i$ leads to a budget deficit of $(n-1)\sum_{i \in N} v_i(k^*(\theta), \theta_i)$.

Proposition

The payment rule of the pivot mechanism satisfies $\sum_{i \in N} t_i(\theta) \leq 0$ *for all* $\theta \in \Theta$.

Proof.

$$\sum_{i \in N} t_i(\theta) = \sum_{i \in N} \left(\sum_{j \neq i} v_j(k^*(\theta), \theta_j) - \sum_{j \neq i} v_j(k^*_{-i}(\theta_{-i}), \theta_j) \right)$$

Since k^*_{-i} is value maximizing for the set of agents $j \neq i$,

$$\sum_{j \neq i} v_j(k^*_{-i}(\theta_{-i}), \theta_j) \geq \sum_{j \neq i} v_j(k^*(\theta), \theta_j) \quad \Longrightarrow \quad \sum_{i \in I} t_i(\theta) \leq 0.$$

□

Example: Allocation of a single indivisible object

Environment: $N = \{1, \ldots, n\}$, $K = \{0, 1, \ldots, n\}$, $\Theta_i = [0, 1]$,

$$\text{and } v_i(k, \theta_i) = \begin{cases} \theta_i & \text{if } k = i, \\ 0 & \text{otherwise.} \end{cases}$$

Example: Allocation of a single indivisible object

Environment: $N = \{1, \ldots, n\}$, $K = \{0, 1, \ldots, n\}$, $\Theta_i = [0, 1]$,

$$\text{and } v_i(k, \theta_i) = \begin{cases} \theta_i & \text{if } k = i, \\ 0 & \text{otherwise.} \end{cases}$$

Pivot mechanism (k^*, t): $\quad k^*(\theta) = \min \left\{ i \in N \,\middle|\, \theta_i = \max_{j \in N} \theta_j \right\},$

$$k^*_{-i}(\theta_{-i}) = \min \left\{ j \in N \setminus i \,\middle|\, \theta_j = \max_{l \in N \setminus i} \theta_l \right\},$$

Example: Allocation of a single indivisible object

Environment: $N = \{1, \ldots, n\}$, $K = \{0, 1, \ldots, n\}$, $\Theta_i = [0, 1]$,

and $v_i(k, \theta_i) = \begin{cases} \theta_i & \text{if } k = i, \\ 0 & \text{otherwise.} \end{cases}$

Pivot mechanism (k^*, t):

$$k^*(\theta) = \min\left\{ i \in N \;\middle|\; \theta_i = \max_{j \in N} \theta_j \right\},$$

$$k^*_{-i}(\theta_{-i}) = \min\left\{ j \in N \setminus i \;\middle|\; \theta_j = \max_{l \in N \setminus i} \theta_l \right\},$$

$$t_i(\theta) = \sum_{j \neq i} v_j(k^*(\theta), \theta_j) - \sum_{j \neq i} v_j(k^*_{-i}(\theta_{-i}), \theta_j)$$

$$= \begin{cases} -\max_{l \in N \setminus i} \theta_l & \text{if } k^*(\theta) = i, \\ 0 & \text{otherwise.} \end{cases}$$

Example: Allocation of a single indivisible object

Environment: $N = \{1, \ldots, n\}$, $K = \{0, 1, \ldots, n\}$, $\Theta_i = [0, 1]$,

$$\text{and } v_i(k, \theta_i) = \begin{cases} \theta_i & \text{if } k = i, \\ 0 & \text{otherwise.} \end{cases}$$

Pivot mechanism (k^*, t):

$$k^*(\theta) = \min\left\{ i \in N \,\middle|\, \theta_i = \max_{j \in N} \theta_j \right\},$$

$$k^*_{-i}(\theta_{-i}) = \min\left\{ j \in N \setminus i \,\middle|\, \theta_j = \max_{l \in N \setminus i} \theta_l \right\},$$

$$t_i(\theta) = \sum\nolimits_{j \neq i} v_j(k^*(\theta), \theta_j) - \sum\nolimits_{j \neq i} v_j(k^*_{-i}(\theta_{-i}), \theta_j)$$

$$= \begin{cases} -\max_{l \in N \setminus i} \theta_l & \text{if } k^*(\theta) = i, \\ 0 & \text{otherwise.} \end{cases}$$

$\Rightarrow$ (k^*, t) is exactly the SCF that is implemented by the **second-price auction**!
(The SPA typically generates a budget surplus that goes to the seller.)

3.3 Budget Balance

Define

$$V(\theta) := \sum_{i \in N} v_i(k^*(\theta), \theta_i)$$

to be the total value generated by a value maximizing allocation rule k^*.

3.3 Budget Balance

Define

$$V(\theta) := \sum_{i \in N} v_i(k^*(\theta), \theta_i)$$

to be the total value generated by a value maximizing allocation rule k^*.

Proposition

A VCG mechanism that satisfies budget balance exists if and only if there are functions $V_1(\theta_{-1}), \ldots, V_n(\theta_{-n})$ such that

$$\sum_{i \in N} V_i(\theta_{-i}) = V(\theta).$$

3.3 Budget Balance

Define

$$V(\theta) := \sum_{i \in N} v_i(k^*(\theta), \theta_i)$$

to be the total value generated by a value maximizing allocation rule k^*.

Proposition

A VCG mechanism that satisfies budget balance exists if and only if there are functions $V_1(\theta_{-1}), \ldots, V_n(\theta_{-n})$ such that

$$\sum_{i \in N} V_i(\theta_{-i}) = V(\theta).$$

You will be asked to prove this result as part of an exercise in the tutorial.

Example: public project

$N = \{1, 2\}$, $K = \{0, 1\}$, $\Theta_i = [-1, 1]$, and $v_i(k, \theta_i) = k\theta_i$.

$$\implies V(\theta) = \max\{0, \theta_1 + \theta_2\} \neq V_1(\theta_2) + V_2(\theta_1).$$

An important special case

Corollary

If $\Theta_i = \{\theta_i\}$, i.e., Θ_i is a singleton, for some $i \in N$, then the ex post efficient SCF is truthfully implementable in dominant strategies.

An important special case

Corollary

If $\Theta_i = \{\theta_i\}$, i.e., Θ_i is a singleton, for some $i \in N$, then the ex post efficient SCF is truthfully implementable in dominant strategies.

Proof.

Set $V_i(\theta_{-i}) = V(\theta)$ and $V_j(\theta_{-j}) = 0$ for all $j \neq i$. □

An important special case

Corollary

If $\Theta_i = \{\theta_i\}$, i.e., Θ_i is a singleton, for some $i \in N$, then the ex post efficient SCF is truthfully implementable in dominant strategies.

Proof.

Set $V_i(\theta_{-i}) = V(\theta)$ and $V_j(\theta_{-j}) = 0$ for all $j \neq i$. □

- If $\Theta_i = \{\theta_i\}$, then agent i has no private information. Then we can use the pivot mechanism to provide incentives for the remaining agents in $N \setminus i$ and transfer the budget surplus to agent i without distorting incentives.

An important special case

Corollary

If $\Theta_i = \{\theta_i\}$, i.e., Θ_i is a singleton, for some $i \in N$, then the ex post efficient SCF is truthfully implementable in dominant strategies.

Proof.

Set $V_i(\theta_{-i}) = V(\theta)$ and $V_j(\theta_{-j}) = 0$ for all $j \neq i$. $\square$

- If $\Theta_i = \{\theta_i\}$, then agent i has no private information. Then we can use the pivot mechanism to provide incentives for the remaining agents in $N \setminus i$ and transfer the budget surplus to agent i without distorting incentives.
- Example: second-price auction where the seller is an agent who has no private information.

Beyond VCG?

- Apart from VCG mechanisms, are there other direct mechanisms that implement the value maximizing allocation rule k^* in dominant strategies?
- The next result identifies a class of environments where this is not the case.

Beyond VCG?

- Apart from VCG mechanisms, are there other direct mechanisms that implement the value maximizing allocation rule k^* in dominant strategies?
- The next result identifies a class of environments where this is not the case.

Let $\mathcal{V}$ denote the set of all possible functions $v\colon K \to \mathbb{R}$.

Proposition

*Suppose that for each agent $i \in N$, $\{v_i(\cdot, \theta_i) \mid \theta_i \in \Theta_i\} = \mathcal{V}$, i.e., every possible value function from K to $\mathbb{R}$ arises for some $\theta_i \in \Theta_i$. Then a SCF $c = (k^*, t)$ with value maximizing allocation rule k^* is truthfully implementable in dominant strategies if and **only if** t is the payment rule of a VCG mechanism.*

For the proof see MWG, p. 879.

3.4 Ex post efficiency without dominant strategies

- As we have seen, there are environments where no ex post efficient SCF can be implemented in **dominant strategies**.
  - In many cases, VCG mechanisms cannot have a balanced budget.
  - But in some environments, VCG mechanisms are the only mechanisms that truthfully implement value maximizing allocations in dominant strategies.

3.4 Ex post efficiency without dominant strategies

- As we have seen, there are environments where no ex post efficient SCF can be implemented in **dominant strategies**.
  - In many cases, VCG mechanisms cannot have a balanced budget.
  - But in some environments, VCG mechanisms are the only mechanisms that truthfully implement value maximizing allocations in dominant strategies.
- However, for *independent private values*, always at least one ex post efficient SCF can be implemented in **Bayesian Nash equilibrium**.

3.4 Ex post efficiency without dominant strategies

- As we have seen, there are environments where no ex post efficient SCF can be implemented in **dominant strategies**.
  - In many cases, VCG mechanisms cannot have a balanced budget.
  - But in some environments, VCG mechanisms are the only mechanisms that truthfully implement value maximizing allocations in dominant strategies.
- However, for *independent private values*, always at least one ex post efficient SCF can be implemented in **Bayesian Nash equilibrium**.

From now on, we assume statistically **independent types**,
i.e., for each agent i, θ_i is independently drawn from some distribution F_i.

The expected externality mechanism

Let k^* be a value maximizing allocation rule.

Define

$$\xi_i(\theta_i) := E_{\theta_{-i}}\Big[\sum_{j\neq i} v_j(k^*(\theta_i,\theta_{-i}),\theta_j)\Big].$$

- $\xi_i(\theta_i)$ represents the expected values of agents $j \neq i$ when i reports θ_i and all $j \neq i$ report truthfully. (Note: ξ_i is a function of only θ_i and *not* of θ_{-i}.)
- The change in ξ_i when agent i changes his report is the *expected externality* of this change on agents $j \neq i$.

The expected externality mechanism

Let k^* be a value maximizing allocation rule.

Define

$$\xi_i(\theta_i) := E_{\theta_{-i}}\Big[\sum_{j\neq i} v_j(k^*(\theta_i,\theta_{-i}),\theta_j)\Big].$$

- $\xi_i(\theta_i)$ represents the expected values of agents $j \neq i$ when i reports θ_i and all $j \neq i$ report truthfully. (Note: ξ_i is a function of only θ_i and *not* of θ_{-i}.)
- The change in ξ_i when agent i changes his report is the *expected externality* of this change on agents $j \neq i$.

Definition (d'Aspremont and Gérard-Varet, 1979; Arrow, 1979)

The **expected externality mechanism** is the direct mechanism $\Gamma = (\Theta_1, \ldots, \Theta_n, (k^*, t))$ where $t_i(\theta) = \xi_i(\theta_i) - \frac{1}{n-1}\sum_{j\neq i}\xi_j(\theta_j)$ for all i.

The following proposition implies that the expected externality mechanism truthfully implements an ex post efficient SCF.

Proposition

The SCF $c = (k^*, t)$ is truthfully implementable in Bayesian Nash equilibrium if

$$t_i(\theta) = \xi_i(\theta_i) + h_i(\theta_{-i}) \quad \text{for all } i \in N, \tag{2}$$

where h_i is an arbitrary function $h_i \colon \Theta_{-i} \to \mathbb{R}$.

The following proposition implies that the expected externality mechanism truthfully implements an ex post efficient SCF.

Proposition

The SCF $c = (k^*, t)$ is truthfully implementable in Bayesian Nash equilibrium if

$$t_i(\theta) = \xi_i(\theta_i) + h_i(\theta_{-i}) \quad \text{for all } i \in N, \tag{2}$$

where h_i is an arbitrary function $h_i \colon \Theta_{-i} \to \mathbb{R}$.
The SCF $c = (k^*, t)$ is ex post efficient if t satisfies (2) and

$$h_i(\theta_{-i}) = -\frac{1}{n-1} \sum_{j \neq i} \xi_j(\theta_j).$$

Proof.

Consider agent i and suppose all agents $j \neq i$ report their types truthfully in the direct mechanism with outcome function $c = (k^*, t)$ where t satisfies (2).
i's expected payoff if he has type θ_i and reports $\hat{\theta}_i$ is

$$E_{\theta_{-i}}\Big[v_i(k^*(\hat{\theta}_i, \theta_{-i}), \theta_i) + t(\hat{\theta}_i, \theta_{-i})\Big]$$
$$= E_{\theta_{-i}}\Big[\sum_{j \in N} v_j(k^*(\hat{\theta}_i, \theta_{-i}), \theta_j)\Big] + E_{\theta_{-i}}[h_i(\theta_{-i})].$$

Proof.

Consider agent i and suppose all agents $j \neq i$ report their types truthfully in the direct mechanism with outcome function $c = (k^*, t)$ where t satisfies (2).
i's expected payoff if he has type θ_i and reports $\hat{\theta}_i$ is

$$E_{\theta_{-i}}\Big[v_i(k^*(\hat{\theta}_i, \theta_{-i}), \theta_i) + t(\hat{\theta}_i, \theta_{-i})\Big]$$
$$= E_{\theta_{-i}}\Big[\sum_{j \in N} v_j(k^*(\hat{\theta}_i, \theta_{-i}), \theta_j)\Big] + E_{\theta_{-i}}[h_i(\theta_{-i})].$$

The first part is maximized at report $\hat{\theta}_i = \theta_i$ since $k = k^*(\theta)$ maximizes $\sum_{j \in N} v_j(k, \theta_j)$. The second part, $E_{\theta_{-i}}[h_i(\theta_{-i})]$, is independent of the report $\hat{\theta}_i$.
$\Longrightarrow$ $\hat{\theta}_i = \theta_i$ is best response of i, i.e., c is truthfully implementable in BNE.

Proof.

Consider agent i and suppose all agents $j \neq i$ report their types truthfully in the direct mechanism with outcome function $c = (k^*, t)$ where t satisfies (2).
i's expected payoff if he has type θ_i and reports $\hat{\theta}_i$ is

$$E_{\theta_{-i}}\Big[v_i(k^*(\hat{\theta}_i, \theta_{-i}), \theta_i) + t(\hat{\theta}_i, \theta_{-i})\Big]$$
$$= E_{\theta_{-i}}\Big[\sum_{j \in N} v_j(k^*(\hat{\theta}_i, \theta_{-i}), \theta_j)\Big] + E_{\theta_{-i}}[h_i(\theta_{-i})].$$

The first part is maximized at report $\hat{\theta}_i = \theta_i$ since $k = k^*(\theta)$ maximizes $\sum_{j \in N} v_j(k, \theta_j)$. The second part, $E_{\theta_{-i}}[h_i(\theta_{-i})]$, is independent of the report $\hat{\theta}_i$.
$\implies$ $\hat{\theta}_i = \theta_i$ is best response of i, i.e., c is truthfully implementable in BNE.

If $h_i(\theta_{-i}) = -\frac{1}{n-1} \sum_{j \neq i} \xi_j(\theta_j)$, t satisfies budget balance since

$$\sum_{i \in N} t_i(\theta) = \sum_{i \in N} \xi_i(\theta_i) - \frac{1}{n-1} \sum_{i \in N} \sum_{j \neq i} \xi_j(\theta_j) = 0.$$

□

Remarks

- Weakening the equilibrium concept to Bayesian Nash equilibrium makes implementation of an ex post efficient SCF possible in general.

  Drawback of Bayesian implementation: payment rule t of the expected externality mechanism depends on type distributions $F_1, \ldots, F_n$.

Remarks

- Weakening the equilibrium concept to Bayesian Nash equilibrium makes implementation of an ex post efficient SCF possible in general.

  Drawback of Bayesian implementation: payment rule t of the expected externality mechanism depends on type distributions $F_1, \ldots, F_n$.

- The expected externality mechanism implements one specific ex post efficient SCF. $\rightarrow$ It results in a particular distribution of utility across agents.

- What other SCFs are implementable in Bayesian Nash equilibrium?
  - There may be additional requirements (e.g. *participation constraints*) that the expected externality mechanism does not satisfy.
  - We may be interested in goals other than ex post efficiency, e.g., maximizing the utility of one agent / revenue.

Remarks

- Weakening the equilibrium concept to Bayesian Nash equilibrium makes implementation of an ex post efficient SCF possible in general.

  Drawback of Bayesian implementation: payment rule t of the expected externality mechanism depends on type distributions $F_1, \ldots, F_n$.

- The expected externality mechanism implements one specific ex post efficient SCF. $\rightarrow$ It results in a particular distribution of utility across agents.

- What other SCFs are implementable in Bayesian Nash equilibrium?
  - There may be additional requirements (e.g. *participation constraints*) that the expected externality mechanism does not satisfy.
  - We may be interested in goals other than ex post efficiency, e.g., maximizing the utility of one agent / revenue.

In the next section, we identify the class of SCFs that are implementable in Bayesian Nash equilibrium if utility functions are linear in types.

4 Bayesian Implementation with Linear Utility

4.1 Characterization of Bayesian implementable SCFs

In this section, we focus on **linear** utilities and **independent private values**.

Assumptions:

- Set of possible types of agent i is an interval:

$$\Theta_i = [\underline{\theta}_i, \overline{\theta}_i] \subset \mathbb{R}, \quad \text{with } \underline{\theta}_i < \overline{\theta}_i.$$

4.1 Characterization of Bayesian implementable SCFs

In this section, we focus on **linear** utilities and **independent private values**.

Assumptions:

- Set of possible types of agent i is an interval:

$$\Theta_i = [\underline{\theta}_i, \overline{\theta}_i] \subset \mathbb{R}, \quad \text{with } \underline{\theta}_i < \overline{\theta}_i.$$

- θ_i is independently drawn from a distribution with probability density f_i satisfying $f_i(\theta_i) > 0$ for all $\theta_i \in [\underline{\theta}_i, \overline{\theta}_i]$.

4.1 Characterization of Bayesian implementable SCFs

In this section, we focus on **linear** utilities and **independent private values**.

Assumptions:

- Set of possible types of agent i is an interval:

$$\Theta_i = [\underline{\theta}_i, \overline{\theta}_i] \subset \mathbb{R}, \quad \text{with } \underline{\theta}_i < \overline{\theta}_i.$$

- θ_i is independently drawn from a distribution with probability density f_i satisfying $f_i(\theta_i) > 0$ for all $\theta_i \in [\underline{\theta}_i, \overline{\theta}_i]$.

- Utility function of agent i:

$$u_i((k,t),\theta_i) = \theta_i v_i(k) + t_i.$$

Definitions

Consider a SCF $c = (k, t)$ and suppose all agents $j \neq i$ truthfully report their types θ_{-i} in the direct mechanism $\Gamma = (\Theta_1, \ldots, \Theta_n, c)$.

Definitions

Consider a SCF $c = (k, t)$ and suppose all agents $j \neq i$ truthfully report their types θ_{-i} in the direct mechanism $\Gamma = (\Theta_1, \ldots, \Theta_n, c)$.

Define for agent i who reports $\hat{\theta}_i$

- the interim **expected allocation**

$$\overline{v}_i(\hat{\theta}_i) := E_{\theta_{-i}}\left[v_i(k(\hat{\theta}_i, \theta_{-i}))\right]$$

- and the interim **expected transfer**

$$\overline{t}_i(\hat{\theta}_i) := E_{\theta_{-i}}\left[t_i(\hat{\theta}_i, \theta_{-i})\right].$$

Definitions

Consider a SCF $c = (k, t)$ and suppose all agents $j \neq i$ truthfully report their types θ_{-i} in the direct mechanism $\Gamma = (\Theta_1, \ldots, \Theta_n, c)$.

Define for agent i who reports $\hat{\theta}_i$

- the interim **expected allocation**

$$\bar{v}_i(\hat{\theta}_i) := E_{\theta_{-i}}\big[v_i(k(\hat{\theta}_i, \theta_{-i}))\big]$$

- and the interim **expected transfer**

$$\bar{t}_i(\hat{\theta}_i) := E_{\theta_{-i}}\big[t_i(\hat{\theta}_i, \theta_{-i})\big].$$

If all agents report truthfully, agent i's interim **expected utility** given his type θ_i is

$$U_i(\theta_i) := E_{\theta_{-i}}\big[u_i((k(\theta), t(\theta)), \theta_i)\big] = \theta_i \bar{v}_i(\theta_i) + \bar{t}_i(\theta_i).$$

Bayesian Incentive Compatibility

A direct mechanism with SCF $c = (k, t)$ is **Bayesian incentive compatible** if and only if truthtelling is a Bayesian Nash equilibrium, i.e.,

$$U_i(\theta_i) \geq \theta_i \bar{v}_i(\hat{\theta}_i) + \bar{t}_i(\hat{\theta}_i) \quad \text{for all } \theta_i, \hat{\theta}_i \in \Theta_i \text{ and } i \in N. \tag{3}$$

Bayesian Incentive Compatibility

A direct mechanism with SCF $c = (k, t)$ is **Bayesian incentive compatible** if and only if truthtelling is a Bayesian Nash equilibrium, i.e.,

$$U_i(\theta_i) \geq \theta_i \overline{v}_i(\hat{\theta}_i) + \overline{t}_i(\hat{\theta}_i) \quad \text{for all } \theta_i, \hat{\theta}_i \in \Theta_i \text{ and } i \in N. \tag{3}$$

Proposition

The direct mechanism with SCF $c = (k, t)$ is Bayesian incentive compatible ***if and only if****, for all $i \in N$,*

1. $\overline{v}_i(\theta_i)$ *is non-decreasing in θ_i,* (IC1)
2. $U_i(\theta_i) = U_i(\underline{\theta}_i) + \int_{\underline{\theta}_i}^{\theta_i} \overline{v}_i(z)dz$ *for all $\theta_i \in \Theta_i$.* (IC2)

Proof: necessity ("only if")

Suppose c is incentive compatible. Then (3) implies, for $\hat{\theta}_i > \theta_i$,

$$U_i(\theta_i) \geq \theta_i \overline{v}_i(\hat{\theta}_i) + \overline{t}_i(\hat{\theta}_i) = U_i(\hat{\theta}_i) + (\theta_i - \hat{\theta}_i)\overline{v}_i(\hat{\theta}_i)$$

$$\text{and } U_i(\hat{\theta}_i) \geq \hat{\theta}_i \overline{v}_i(\theta_i) + \overline{t}_i(\theta_i) = U_i(\theta_i) + (\hat{\theta}_i - \theta_i)\overline{v}_i(\theta_i).$$

Proof: necessity ("only if")

Suppose c is incentive compatible. Then (3) implies, for $\hat{\theta}_i > \theta_i$,

$$U_i(\theta_i) \geq \theta_i \overline{v}_i(\hat{\theta}_i) + \overline{t}_i(\hat{\theta}_i) = U_i(\hat{\theta}_i) + (\theta_i - \hat{\theta}_i)\overline{v}_i(\hat{\theta}_i)$$

$$\text{and } U_i(\hat{\theta}_i) \geq \hat{\theta}_i \overline{v}_i(\theta_i) + \overline{t}_i(\theta_i) = U_i(\theta_i) + (\hat{\theta}_i - \theta_i)\overline{v}_i(\theta_i).$$

$$\implies \overline{v}_i(\hat{\theta}_i) \geq \frac{U_i(\hat{\theta}_i) - U_i(\theta_i)}{\hat{\theta}_i - \theta_i} \geq \overline{v}_i(\theta_i). \tag{4}$$

Proof: necessity ("only if")

Suppose c is incentive compatible. Then (3) implies, for $\hat{\theta}_i > \theta_i$,

$$U_i(\theta_i) \geq \theta_i \overline{v}_i(\hat{\theta}_i) + \overline{t}_i(\hat{\theta}_i) = U_i(\hat{\theta}_i) + (\theta_i - \hat{\theta}_i)\overline{v}_i(\hat{\theta}_i)$$

$$\text{and } U_i(\hat{\theta}_i) \geq \hat{\theta}_i \overline{v}_i(\theta_i) + \overline{t}_i(\theta_i) = U_i(\theta_i) + (\hat{\theta}_i - \theta_i)\overline{v}_i(\theta_i).$$

$$\implies \overline{v}_i(\hat{\theta}_i) \geq \frac{U_i(\hat{\theta}_i) - U_i(\theta_i)}{\hat{\theta}_i - \theta_i} \geq \overline{v}_i(\theta_i). \tag{4}$$

$\implies \overline{v}_i(\theta_i)$ is non-decreasing in θ_i.

Proof: necessity ("only if")

Suppose c is incentive compatible. Then (3) implies, for $\hat{\theta}_i > \theta_i$,

$$\begin{aligned} U_i(\theta_i) &\geq \theta_i \overline{v}_i(\hat{\theta}_i) + \overline{t}_i(\hat{\theta}_i) = U_i(\hat{\theta}_i) + (\theta_i - \hat{\theta}_i)\overline{v}_i(\hat{\theta}_i) \\ \text{and } U_i(\hat{\theta}_i) &\geq \hat{\theta}_i \overline{v}_i(\theta_i) + \overline{t}_i(\theta_i) = U_i(\theta_i) + (\hat{\theta}_i - \theta_i)\overline{v}_i(\theta_i). \end{aligned}$$

$$\implies \overline{v}_i(\hat{\theta}_i) \geq \frac{U_i(\hat{\theta}_i) - U_i(\theta_i)}{\hat{\theta}_i - \theta_i} \geq \overline{v}_i(\theta_i). \tag{4}$$

$\implies \overline{v}_i(\theta_i)$ is non-decreasing in θ_i.

According to (3), U_i is the maximum of a family of affine functions.

Proof: necessity ("only if")

Suppose c is incentive compatible. Then (3) implies, for $\hat{\theta}_i > \theta_i$,

$$U_i(\theta_i) \geq \theta_i \overline{v}_i(\hat{\theta}_i) + \overline{t}_i(\hat{\theta}_i) = U_i(\hat{\theta}_i) + (\theta_i - \hat{\theta}_i)\overline{v}_i(\hat{\theta}_i)$$

$$\text{and } U_i(\hat{\theta}_i) \geq \hat{\theta}_i \overline{v}_i(\theta_i) + \overline{t}_i(\theta_i) = U_i(\theta_i) + (\hat{\theta}_i - \theta_i)\overline{v}_i(\theta_i).$$

$$\implies \overline{v}_i(\hat{\theta}_i) \geq \frac{U_i(\hat{\theta}_i) - U_i(\theta_i)}{\hat{\theta}_i - \theta_i} \geq \overline{v}_i(\theta_i). \tag{4}$$

$\implies$ $\overline{v}_i(\theta_i)$ is non-decreasing in θ_i.

According to (3), U_i is the maximum of a family of affine functions.
$\implies$ U_i is a convex function.

Proof: necessity ("only if")

Suppose c is incentive compatible. Then (3) implies, for $\hat{\theta}_i > \theta_i$,

$$U_i(\theta_i) \geq \theta_i \overline{v}_i(\hat{\theta}_i) + \overline{t}_i(\hat{\theta}_i) = U_i(\hat{\theta}_i) + (\theta_i - \hat{\theta}_i)\overline{v}_i(\hat{\theta}_i)$$

$$\text{and } U_i(\hat{\theta}_i) \geq \hat{\theta}_i \overline{v}_i(\theta_i) + \overline{t}_i(\theta_i) = U_i(\theta_i) + (\hat{\theta}_i - \theta_i)\overline{v}_i(\theta_i).$$

$$\implies \overline{v}_i(\hat{\theta}_i) \geq \frac{U_i(\hat{\theta}_i) - U_i(\theta_i)}{\hat{\theta}_i - \theta_i} \geq \overline{v}_i(\theta_i). \tag{4}$$

$\implies \overline{v}_i(\theta_i)$ is non-decreasing in θ_i.

According to (3), U_i is the maximum of a family of affine functions.
$\implies U_i$ is a convex function. $\implies U_i$ is absolutely continuous.

Proof: necessity ("only if")

Suppose c is incentive compatible. Then (3) implies, for $\hat{\theta}_i > \theta_i$,

$$\begin{aligned} U_i(\theta_i) &\geq \theta_i \overline{v}_i(\hat{\theta}_i) + \overline{t}_i(\hat{\theta}_i) = U_i(\hat{\theta}_i) + (\theta_i - \hat{\theta}_i)\overline{v}_i(\hat{\theta}_i) \\ \text{and } U_i(\hat{\theta}_i) &\geq \hat{\theta}_i \overline{v}_i(\theta_i) + \overline{t}_i(\theta_i) = U_i(\theta_i) + (\hat{\theta}_i - \theta_i)\overline{v}_i(\theta_i). \end{aligned}$$

$$\Longrightarrow \overline{v}_i(\hat{\theta}_i) \geq \frac{U_i(\hat{\theta}_i) - U_i(\theta_i)}{\hat{\theta}_i - \theta_i} \geq \overline{v}_i(\theta_i). \tag{4}$$

$\Longrightarrow$ $\overline{v}_i(\theta_i)$ is non-decreasing in θ_i.

According to (3), U_i is the maximum of a family of affine functions.
$\Longrightarrow$ U_i is a convex function. $\Longrightarrow$ U_i is absolutely continuous.
$\Longrightarrow$ U_i is differentiable almost everywhere
and equal to the integral of its derivative.

Proof: necessity ("only if")

Suppose c is incentive compatible. Then (3) implies, for $\hat{\theta}_i > \theta_i$,

$$U_i(\theta_i) \geq \theta_i \overline{v}_i(\hat{\theta}_i) + \overline{t}_i(\hat{\theta}_i) = U_i(\hat{\theta}_i) + (\theta_i - \hat{\theta}_i)\overline{v}_i(\hat{\theta}_i)$$

$$\text{and } U_i(\hat{\theta}_i) \geq \hat{\theta}_i \overline{v}_i(\theta_i) + \overline{t}_i(\theta_i) = U_i(\theta_i) + (\hat{\theta}_i - \theta_i)\overline{v}_i(\theta_i).$$

$$\implies \overline{v}_i(\hat{\theta}_i) \geq \frac{U_i(\hat{\theta}_i) - U_i(\theta_i)}{\hat{\theta}_i - \theta_i} \geq \overline{v}_i(\theta_i). \tag{4}$$

$\implies$ $\overline{v}_i(\theta_i)$ is non-decreasing in θ_i.

According to (3), U_i is the maximum of a family of affine functions.
$\implies$ U_i is a convex function. $\implies$ U_i is absolutely continuous.
$\implies$ U_i is differentiable almost everywhere
and equal to the integral of its derivative.

Letting $\hat{\theta}_i \to \theta_i$ in (4) yields $U_i'(\theta_i) = \overline{v}_i(\theta_i)$ for almost all θ_i

Proof: necessity ("only if")

Suppose c is incentive compatible. Then (3) implies, for $\hat{\theta}_i > \theta_i$,

$$U_i(\theta_i) \geq \theta_i \overline{v}_i(\hat{\theta}_i) + \overline{t}_i(\hat{\theta}_i) = U_i(\hat{\theta}_i) + (\theta_i - \hat{\theta}_i)\overline{v}_i(\hat{\theta}_i)$$

$$\text{and } U_i(\hat{\theta}_i) \geq \hat{\theta}_i \overline{v}_i(\theta_i) + \overline{t}_i(\theta_i) = U_i(\theta_i) + (\hat{\theta}_i - \theta_i)\overline{v}_i(\theta_i).$$

$$\implies \overline{v}_i(\hat{\theta}_i) \geq \frac{U_i(\hat{\theta}_i) - U_i(\theta_i)}{\hat{\theta}_i - \theta_i} \geq \overline{v}_i(\theta_i). \tag{4}$$

$\implies$ $\overline{v}_i(\theta_i)$ is non-decreasing in θ_i.

According to (3), U_i is the maximum of a family of affine functions.

$\implies$ U_i is a convex function. $\implies$ U_i is absolutely continuous.

$\implies$ U_i is differentiable almost everywhere and equal to the integral of its derivative.

Letting $\hat{\theta}_i \to \theta_i$ in (4) yields $U_i'(\theta_i) = \overline{v}_i(\theta_i)$ for almost all θ_i

and therefore $U_i(\theta_i) - U_i(\underline{\theta}_i) = \int_{\underline{\theta}_i}^{\theta_i} \overline{v}_i(z)\,dz.$ □

Proof: sufficiency ("if")

Consider any θ_i and $\hat{\theta}_i$ and suppose (IC1) and (IC2) hold. Hence,

$$U_i(\theta_i) - U_i(\hat{\theta}_i) = \int_{\hat{\theta}_i}^{\theta_i} \overline{v}_i(z)\,dz.$$

Proof: sufficiency ("if")

Consider any θ_i and $\hat{\theta}_i$ and suppose (IC1) and (IC2) hold. Hence,

$$U_i(\theta_i) - U_i(\hat{\theta}_i) = \int_{\hat{\theta}_i}^{\theta_i} \overline{v}_i(z)\,dz.$$

If $\theta_i > \hat{\theta}_i$,

$$\begin{aligned} U_i(\theta_i) &= U_i(\hat{\theta}_i) + \int_{\hat{\theta}_i}^{\theta_i} \overline{v}_i(z)\,dz \\ &\geq U_i(\hat{\theta}_i) + \int_{\hat{\theta}_i}^{\theta_i} \overline{v}_i(\hat{\theta}_i)\,dz = U_i(\hat{\theta}_i) + (\theta_i - \hat{\theta}_i)\overline{v}_i(\hat{\theta}_i) = \theta_i \overline{v}_i(\hat{\theta}_i) + \overline{t}_i(\hat{\theta}_i). \end{aligned}$$

Proof: sufficiency ("if")

Consider any θ_i and $\hat{\theta}_i$ and suppose (IC1) and (IC2) hold. Hence,

$$U_i(\theta_i) - U_i(\hat{\theta}_i) = \int_{\hat{\theta}_i}^{\theta_i} \overline{v}_i(z)\,dz.$$

If $\theta_i > \hat{\theta}_i$,

$$\begin{aligned} U_i(\theta_i) &= U_i(\hat{\theta}_i) + \int_{\hat{\theta}_i}^{\theta_i} \overline{v}_i(z)\,dz \\ &\geq U_i(\hat{\theta}_i) + \int_{\hat{\theta}_i}^{\theta_i} \overline{v}_i(\hat{\theta}_i)\,dz = U_i(\hat{\theta}_i) + (\theta_i - \hat{\theta}_i)\overline{v}_i(\hat{\theta}_i) = \theta_i \overline{v}_i(\hat{\theta}_i) + \overline{t}_i(\hat{\theta}_i). \end{aligned}$$

If $\theta_i < \hat{\theta}_i$,

$$\begin{aligned} U_i(\theta_i) &= U_i(\hat{\theta}_i) - \int_{\theta_i}^{\hat{\theta}_i} \overline{v}_i(z)\,dz \\ &\geq U_i(\hat{\theta}_i) - \int_{\theta_i}^{\hat{\theta}_i} \overline{v}_i(\hat{\theta}_i)\,dz = U_i(\hat{\theta}_i) - (\hat{\theta}_i - \theta_i)\overline{v}_i(\hat{\theta}_i) = \theta_i \overline{v}_i(\hat{\theta}_i) + \overline{t}_i(\hat{\theta}_i). \end{aligned}$$

Proof: sufficiency ("if")

Consider any θ_i and $\hat{\theta}_i$ and suppose (IC1) and (IC2) hold. Hence,

$$U_i(\theta_i) - U_i(\hat{\theta}_i) = \int_{\hat{\theta}_i}^{\theta_i} \overline{v}_i(z)\,dz.$$

If $\theta_i > \hat{\theta}_i$,

$$\begin{aligned} U_i(\theta_i) &= U_i(\hat{\theta}_i) + \int_{\hat{\theta}_i}^{\theta_i} \overline{v}_i(z)\,dz \\ &\geq U_i(\hat{\theta}_i) + \int_{\hat{\theta}_i}^{\theta_i} \overline{v}_i(\hat{\theta}_i)\,dz = U_i(\hat{\theta}_i) + (\theta_i - \hat{\theta}_i)\overline{v}_i(\hat{\theta}_i) = \theta_i \overline{v}_i(\hat{\theta}_i) + \overline{t}_i(\hat{\theta}_i). \end{aligned}$$

If $\theta_i < \hat{\theta}_i$,

$$\begin{aligned} U_i(\theta_i) &= U_i(\hat{\theta}_i) - \int_{\theta_i}^{\hat{\theta}_i} \overline{v}_i(z)\,dz \\ &\geq U_i(\hat{\theta}_i) - \int_{\theta_i}^{\hat{\theta}_i} \overline{v}_i(\hat{\theta}_i)\,dz = U_i(\hat{\theta}_i) - (\hat{\theta}_i - \theta_i)\overline{v}_i(\hat{\theta}_i) = \theta_i \overline{v}_i(\hat{\theta}_i) + \overline{t}_i(\hat{\theta}_i). \end{aligned}$$

Therefore, c is incentive compatible as (3) is satisfied for all θ_i and $\hat{\theta}_i$. $\square$

Implications for SCFs

The proposition identifies the class of SCFs that are Bayesian implementable:

- The allocation rule k has to satisfy (IC1): each $\overline{v}_i$ has to be non-decreasing. (This requirement cannot be relaxed through the payment rule.)

Implications for SCFs

The proposition identifies the class of SCFs that are Bayesian implementable:

- The allocation rule k has to satisfy (IC1): each $\overline{v}_i$ has to be non-decreasing. (This requirement cannot be relaxed through the payment rule.)

- Given such an allocation rule k, (IC2) pins down interim expected payoffs and transfers up to a constant ($U_i(\underline{\theta}_i)$ and $\overline{t}_i(\underline{\theta}_i)$, respectively):

$$U_i(\theta_i) = U_i(\underline{\theta}_i) + \int_{\underline{\theta}_i}^{\theta_i} \overline{v}_i(z)\, dz \qquad \text{(“\textbf{payoff equivalence}”)}$$

$$\Longrightarrow \; \overline{t}_i(\theta_i) = \overline{t}_i(\underline{\theta}_i) + \underline{\theta}_i \overline{v}_i(\underline{\theta}_i) - \theta_i \overline{v}_i(\theta_i) + \int_{\underline{\theta}_i}^{\theta_i} \overline{v}_i(z)\, dz \quad \text{for all } i \text{ and } \theta_i.$$

Implications for SCFs

The proposition identifies the class of SCFs that are Bayesian implementable:

- The allocation rule k has to satisfy (IC1): each $\overline{v}_i$ has to be non-decreasing. (This requirement cannot be relaxed through the payment rule.)

- Given such an allocation rule k, (IC2) pins down interim expected payoffs and transfers up to a constant ($U_i(\underline{\theta}_i)$ and $\overline{t}_i(\underline{\theta}_i)$, respectively):

$$U_i(\theta_i) = U_i(\underline{\theta}_i) + \int_{\underline{\theta}_i}^{\theta_i} \overline{v}_i(z)\,dz \qquad (\text{ “}\textbf{payoff equivalence}\text{” })$$

$$\Longrightarrow \; \overline{t}_i(\theta_i) = \overline{t}_i(\underline{\theta}_i) + \underline{\theta}_i\overline{v}_i(\underline{\theta}_i) - \theta_i\overline{v}_i(\theta_i) + \int_{\underline{\theta}_i}^{\theta_i} \overline{v}_i(z)\,dz \quad \text{for all } i \text{ and } \theta_i.$$

- Given k and $\overline{t}_i(\underline{\theta}_i)$, the payment rule t has to be such that

$$E_{\theta_{-i}}[t_i(\theta_i, \theta_{-i})] = \overline{t}_i(\theta_i) \quad \text{for all } i \text{ and } \theta_i.$$

(Typically, there are many such t.)

4.2 Application: Revenue Equivalence of Standard Auctions

Auction environment with *symmetric* independent private values:

- single indivisible object, n bidders
- Bidder i has valuation $\theta_i \in [\underline{\theta}, \overline{\theta}]$ for the object.
- Each θ_i is independently drawn from the same distribution (i.e., $f_i = f \, \forall i$).

4.2 Application: Revenue Equivalence of Standard Auctions

Auction environment with *symmetric* independent private values:

- single indivisible object, n bidders
- Bidder i has valuation $\theta_i \in [\underline{\theta}, \overline{\theta}]$ for the object.
- Each θ_i is independently drawn from the same distribution (i.e., $f_i = f \,\forall i$).
- Set of possible allocations: $K = \{1, 2, \ldots, n\}$ where $k = i$ denotes that bidder i obtains the object.
- Bidder i's payoff from alternative (k, t):

$$u_i((k,t),\theta_i) = \theta_i v_i(k) + t \quad \text{with } v_i(k) = \begin{cases} 1 & \text{if } k = i, \\ 0 & \text{otherwise.} \end{cases}$$

A standard auction

Consider an (anonymous) auction that

1. assigns the object to the **highest bidder**,
2. has a **symmetric** Bayesian Nash equilibrium with strictly **increasing** bidding strategy $\beta\colon [\underline{\theta}, \overline{\theta}] \to \mathbb{R}_+$.

A standard auction

Consider an (anonymous) auction that

1. assigns the object to the **highest bidder**,
2. has a **symmetric** Bayesian Nash equilibrium
   with strictly **increasing** bidding strategy $\beta\colon [\underline{\theta}, \overline{\theta}] \to \mathbb{R}_+$.

$\Longrightarrow$ in equilibrium, bidder i with the highest valuation θ_i obtains object.

$\Longrightarrow$ auction implements a SCF (k^*, t), where k^* is value maximizing,
i.e., $k^*(\theta) = i$ if $\theta_i > \max_{j \neq i} \theta_j$.

A standard auction

Consider an (anonymous) auction that

1. assigns the object to the **highest bidder**,
2. has a **symmetric** Bayesian Nash equilibrium
   with strictly **increasing** bidding strategy $\beta\colon [\underline{\theta}, \overline{\theta}] \to \mathbb{R}_+$.

$\implies$ in equilibrium, bidder i with the highest valuation θ_i obtains object.

$\implies$ auction implements a SCF (k^*, t), where k^* is value maximizing, i.e., $k^*(\theta) = i$ if $\theta_i > \max_{j \neq i} \theta_j$.

By the *Revelation Principle*, the direct mechanism with SCF (k^*, t) is Bayesian incentive compatible. $\implies$ (IC2) holds:

$$\theta_i \overline{v}_i(\theta_i) + \overline{t}_i(\theta_i) = \underline{\theta}\,\overline{v}_i(\underline{\theta}) + \overline{t}_i(\underline{\theta}) + \int_{\underline{\theta}}^{\theta_i} \overline{v}_i(z)\,dz$$

Since $\overline{v}_i(\theta_i) = \mathrm{Pr}_{\theta_{-i}}[\max_{j \neq i} \theta_j < \theta_i] = F(\theta_i)^{n-1}$, (IC2) implies

$$\overline{t}_i(\theta_i) = -\theta_i F(\theta_i)^{n-1} + \int_{\underline{\theta}}^{\theta_i} F(z)^{n-1} dz + \overline{t}_i(\underline{\theta})$$

where $\overline{t}_i(\underline{\theta}) = \overline{t}_j(\underline{\theta})$ for all i, j because the auction equilibrium is symmetric.

Since $\overline{v}_i(\theta_i) = \Pr_{\theta_{-i}}[\max_{j\neq i}\theta_j < \theta_i] = F(\theta_i)^{n-1}$, (IC2) implies

$$\overline{t}_i(\theta_i) = -\theta_i F(\theta_i)^{n-1} + \int_{\underline{\theta}}^{\theta_i} F(z)^{n-1}\,dz + \overline{t}_i(\underline{\theta})$$

where $\overline{t}_i(\underline{\theta}) = \overline{t}_j(\underline{\theta})$ for all i, j because the auction equilibrium is symmetric.

Seller's (ex ante) **expected revenue**:

$$\begin{aligned}\sum_{i\in N} E_\theta[-t_i(\theta)] &= \sum_{i\in N} E_{\theta_i}\big[-\overline{t}_i(\theta_i)\big] \\ &= -nE_{\theta_i}\big[\overline{t}_i(\theta_i)\big] \\ &= n\int_{\underline{\theta}}^{\overline{\theta}}\left(yF(y)^{n-1} - \int_{\underline{\theta}}^{y} F(z)^{n-1}\,dz\right) f(y)\,dy - n\overline{t}_i(\underline{\theta}).\end{aligned}$$

Revenue Equivalence

Corollary (Revenue Equivalence Theorem)

Consider any two auctions that award the object to the highest bid and have a symmetric Bayesian Nash equilibrium with a strictly increasing bidding strategy. If the interim expected payment by a bidder with valuation $\underline{\theta}$ is the same in this equilibrium of both auctions, then

- *the **interim expected payment** from each type of each bidder*
- *and therefore the **expected revenue** for the seller*

are the same in both auctions.

Example: second-price vs. first-price auction

- In the SPA (FPA) the highest bidder wins paying the 2nd-highest (his) bid.
  - The SPA has a symmetric BNE with bidding strategy $\beta_S(\theta_i) = \theta_i$.
  - One can show: the FPA has a symmetric BNE with strictly increasing $\beta_F(\cdot)$.
- In both auctions, only the winner makes a payment and type $\underline{\theta}_i$ wins with probability zero: $\bar{t}_i(\underline{\theta}) = 0$.

Example: second-price vs. first-price auction

- In the SPA (FPA) the highest bidder wins paying the 2nd-highest (his) bid.
  - The SPA has a symmetric BNE with bidding strategy $\beta_S(\theta_i) = \theta_i$.
  - One can show: the FPA has a symmetric BNE with strictly increasing $\beta_F(\cdot)$.
- In both auctions, only the winner makes a payment and type $\underline{\theta}_i$ wins with probability zero: $\bar{t}_i(\underline{\theta}) = 0$.

$\Longrightarrow$ Revenue Equivalence: SPA & FPA yield same expected revenue for seller!

Example: second-price vs. first-price auction

- In the SPA (FPA) the highest bidder wins paying the 2nd-highest (his) bid.
  - The SPA has a symmetric BNE with bidding strategy $\beta_S(\theta_i) = \theta_i$.
  - One can show: the FPA has a symmetric BNE with strictly increasing $\beta_F(\cdot)$.
- In both auctions, only the winner makes a payment and type $\underline{\theta}_i$ wins with probability zero: $\bar{t}_i(\underline{\theta}) = 0$.

$\Longrightarrow$ Revenue Equivalence: SPA & FPA yield same expected revenue for seller!

We can use Revenue Equivalence to determine the symmetric BNE of the FPA:

$$-\bar{t}_i^S(\theta_i) = F(\theta_i)^{n-1} E_{\theta_{-i}}\left[\max_{j\neq i} \theta_j \,\middle|\, \max_{j\neq i} \theta_j < \theta_i\right]$$

$$-\bar{t}_i^F(\theta_i) = F(\theta_i)^{n-1} \beta_F(\theta_i)$$

Example: second-price vs. first-price auction

- In the SPA (FPA) the highest bidder wins paying the 2nd-highest (his) bid.
  - The SPA has a symmetric BNE with bidding strategy $\beta_S(\theta_i) = \theta_i$.
  - One can show: the FPA has a symmetric BNE with strictly increasing $\beta_F(\cdot)$.
- In both auctions, only the winner makes a payment and type $\underline{\theta}_i$ wins with probability zero: $\bar{t}_i(\underline{\theta}) = 0$.

$\Longrightarrow$ Revenue Equivalence: SPA & FPA yield same expected revenue for seller!

We can use Revenue Equivalence to determine the symmetric BNE of the FPA:

$$-\bar{t}_i^S(\theta_i) = F(\theta_i)^{n-1} E_{\theta_{-i}}\left[\max_{j\neq i} \theta_j \middle| \max_{j\neq i} \theta_j < \theta_i\right]$$

$$-\bar{t}_i^F(\theta_i) = F(\theta_i)^{n-1} \beta_F(\theta_i)$$

$$\Longrightarrow \quad \beta_F(\theta_i) = E_{\theta_{-i}}\left[\max_{j\neq i} \theta_j \middle| \max_{j\neq i} \theta_j < \theta_i\right]$$

4.3 Participation Constraints

- Up to this point, we have implicitly assumed that agents can be forced to participate in a mechanism.
  - Agents could choose optimal actions within those allowed by the mechanism
  - but agents could not choose whether or not to participate.
- In many application, however, participation is *voluntary*.
  - e.g. a seller owning an object may not be forced to trade it (property rights)

4.3 Participation Constraints

- Up to this point, we have implicitly assumed that agents can be forced to participate in a mechanism.
  - Agents could choose optimal actions within those allowed by the mechanism
  - but agents could not choose whether or not to participate.
- In many application, however, participation is *voluntary*.
  - e.g. a seller owning an object may not be forced to trade it (property rights)

Suppose each agent i has an outside option:

Agent i can obtain utility $\hat{u}_i(\theta_i)$ by withdrawing from the mechanism.

A mechanism is **individually rational** for agent i if it is optimal for i to participate.

Types of Individual Rationality (IR)

Depending on the stage at which an agent makes the participation decision, we distinguish three types of individual rationality.

A direct mechanism with SCF $c = (k, t)$ satisfies for agent i

- **ex post individual rationality** if

$$u_i((k(\theta), t(\theta)), \theta_i) \geq \hat{u}_i(\theta_i) \quad \text{for all } \theta \in \Theta.$$

Types of Individual Rationality (IR)

Depending on the stage at which an agent makes the participation decision, we distinguish three types of individual rationality.

A direct mechanism with SCF $c = (k, t)$ satisfies for agent i

- **ex post individual rationality** if

$$u_i((k(\theta), t(\theta)), \theta_i) \geq \hat{u}_i(\theta_i) \quad \text{for all } \theta \in \Theta.$$

- **interim individual rationality** if

$$U_i(\theta_i) = E_{\theta_{-i}}\left[u_i((k(\theta), t(\theta)), \theta_i)\right] \geq \hat{u}_i(\theta_i) \quad \text{for all } \theta_i \in \Theta_i.$$

Types of Individual Rationality (IR)

Depending on the stage at which an agent makes the participation decision, we distinguish three types of individual rationality.

A direct mechanism with SCF $c = (k, t)$ satisfies for agent i

- **ex post individual rationality** if

$$u_i((k(\theta), t(\theta)), \theta_i) \geq \hat{u}_i(\theta_i) \quad \text{for all } \theta \in \Theta.$$

- **interim individual rationality** if

$$U_i(\theta_i) = E_{\theta_{-i}}\big[u_i((k(\theta), t(\theta)), \theta_i)\big] \geq \hat{u}_i(\theta_i) \quad \text{for all } \theta_i \in \Theta_i.$$

- **ex ante individual rationality** if

$$E[U_i(\theta_i)] = E_\theta\big[u_i((k(\theta), t(\theta)), \theta_i)\big] \geq E[\hat{u}_i(\theta_i)].$$

Remarks

Note that

$$\text{ex post IR} \quad \Longrightarrow \quad \text{interim IR} \quad \Longrightarrow \quad \text{ex ante IR}$$

but the reverse is not true.

Remarks

Note that

$$\text{ex post IR} \quad \Longrightarrow \quad \text{interim IR} \quad \Longrightarrow \quad \text{ex ante IR}$$

but the reverse is not true.

- Ex post IR imposes the most severe constraint, ex ante IR the least severe.
- Which type of individual rationality constraint is relevant, depends on the particular application we study.

Remarks

Note that

$$\text{ex post IR} \quad \Longrightarrow \quad \text{interim IR} \quad \Longrightarrow \quad \text{ex ante IR}$$

but the reverse is not true.

- Ex post IR imposes the most severe constraint, ex ante IR the least severe.
- Which type of individual rationality constraint is relevant, depends on the particular application we study.

In general, private information may restrict the set of implementable SCFs not only through incentive compatibility constraints, but also through individual rationality constraints.

5 Bilateral Trade

(Myerson and Satterthwaite, 1983)

5.1 Setup

One seller S who owns a single indivisible object, one potential buyer B.

- S has valuation for (or cost of producing) the object $\theta_S \in [\underline{\theta}_S, \overline{\theta}_S]$.
- B has valuation $\theta_B \in [\underline{\theta}_B, \overline{\theta}_B]$.
- Each θ_i is independently drawn from a continuous distribution with density $f_i(\theta_i) > 0$ for all $\theta_i \in [\underline{\theta}_i, \overline{\theta}_i]$.

5.1 Setup

One seller S who owns a single indivisible object, one potential buyer B.

- S has valuation for (or cost of producing) the object $\theta_S \in [\underline{\theta}_S, \overline{\theta}_S]$.
- B has valuation $\theta_B \in [\underline{\theta}_B, \overline{\theta}_B]$.
- Each θ_i is independently drawn from a continuous distribution with density $f_i(\theta_i) > 0$ for all $\theta_i \in [\underline{\theta}_i, \overline{\theta}_i]$.
- $\underline{\theta}_S < \overline{\theta}_B$ and $\underline{\theta}_B < \overline{\theta}_S$.

  $\Rightarrow$ Both $\theta_S < \theta_B$ and $\theta_S > \theta_B$ happen with strictly positive probability.

5.1 Setup

One seller S who owns a single indivisible object, one potential buyer B.

- S has valuation for (or cost of producing) the object $\theta_S \in [\underline{\theta}_S, \overline{\theta}_S]$.
- B has valuation $\theta_B \in [\underline{\theta}_B, \overline{\theta}_B]$.
- Each θ_i is independently drawn from a continuous distribution with density $f_i(\theta_i) > 0$ for all $\theta_i \in [\underline{\theta}_i, \overline{\theta}_i]$.
- $\underline{\theta}_S < \overline{\theta}_B$ and $\underline{\theta}_B < \overline{\theta}_S$.

  $\Rightarrow$ Both $\theta_S < \theta_B$ and $\theta_S > \theta_B$ happen with strictly positive probability.

Set of possible allocations $K = [0, 1]$.

- With probability $p \in K$ there is trade: the object is transferred from S to B.

Utility functions:

$$u_S((p,t), \theta_S) = t_S - p\theta_S \qquad \text{(i.e., } v_S(k) = -k\text{)},$$
$$u_B((p,t), \theta_B) = p\theta_B + t_B \qquad \text{(i.e., } v_B(k) = k\text{)}.$$

Social choice functions, ex post efficiency

A social choice function consists of

- an allocation rule $p\colon [\underline{\theta}_S, \overline{\theta}_S] \times [\underline{\theta}_B, \overline{\theta}_B] \to [0,1]$,
- and a payment rule $t = (t_S, t_B)$, where $t_i\colon [\underline{\theta}_S, \overline{\theta}_S] \times [\underline{\theta}_B, \overline{\theta}_B] \to \mathbb{R}$.

Social choice functions, ex post efficiency

A social choice function consists of

- an allocation rule $p\colon [\underline{\theta}_S, \overline{\theta}_S] \times [\underline{\theta}_B, \overline{\theta}_B] \to [0,1]$,
- and a payment rule $t = (t_S, t_B)$, where $t_i\colon [\underline{\theta}_S, \overline{\theta}_S] \times [\underline{\theta}_B, \overline{\theta}_B] \to \mathbb{R}$.

A SCF (p, t) is **ex post efficient** if

- p is a **value** (gains from trade) **maximizing** allocation rule

$$p^*(\cdot) \in \arg\max_{p(\cdot)} (\theta_B - \theta_S) p(\theta_S, \theta_B).$$

$$\implies p^*(\theta_S, \theta_B) = \begin{cases} 1 & \text{if } \theta_S < \theta_B, \\ 0 & \text{if } \theta_S > \theta_B. \end{cases}$$

- transfers satisfy **budget balance**: $t_S(\theta_S, \theta_B) = -t_B(\theta_S, \theta_B)$.

Incentive compatibility and individual rationality

Let $\overline{p}_S(\theta_S) := E_{\theta_B}[p(\theta_S, \theta_B)]$ and $\overline{p}_B(\theta_B) := E_{\theta_S}[p(\theta_S, \theta_B)]$.

Hence, $U_S(\theta_S) = \bar{t}_S(\theta_S) - \overline{p}_S(\theta_S)\theta_S$ and $U_B(\theta_B) = \overline{p}_B(\theta_B)\theta_B + \bar{t}_B(\theta_B)$.

Incentive compatibility and individual rationality

Let $\overline{p}_S(\theta_S) := E_{\theta_B}[p(\theta_S, \theta_B)]$ and $\overline{p}_B(\theta_B) := E_{\theta_S}[p(\theta_S, \theta_B)]$.
Hence, $U_S(\theta_S) = \overline{t}_S(\theta_S) - \overline{p}_S(\theta_S)\theta_S$ and $U_B(\theta_B) = \overline{p}_B(\theta_B)\theta_B + \overline{t}_B(\theta_B)$.

Bayesian incentive compatibility is equivalent to (IC1) and (IC2).

1. (IC1): $\overline{p}_S(\theta_S)$ is *non-increasing* and $\overline{p}_B(\theta_B)$ is *non-decreasing*.

2. (IC2):

$$U_S(\theta_S) = U_S(\underline{\theta}_S) - \int_{\underline{\theta}_S}^{\theta_S} \overline{p}_S(z)\,dz, \quad U_B(\theta_B) = U_B(\underline{\theta}_B) + \int_{\underline{\theta}_B}^{\theta_B} \overline{p}_B(z)\,dz.$$

Incentive compatibility and individual rationality

Let $\overline{p}_S(\theta_S) := E_{\theta_B}[p(\theta_S, \theta_B)]$ and $\overline{p}_B(\theta_B) := E_{\theta_S}[p(\theta_S, \theta_B)]$.
Hence, $U_S(\theta_S) = \overline{t}_S(\theta_S) - \overline{p}_S(\theta_S)\theta_S$ and $U_B(\theta_B) = \overline{p}_B(\theta_B)\theta_B + \overline{t}_B(\theta_B)$.

Bayesian incentive compatibility is equivalent to (IC1) and (IC2).

1. (IC1): $\overline{p}_S(\theta_S)$ is *non-increasing* and $\overline{p}_B(\theta_B)$ is *non-decreasing*.

2. (IC2):
$$U_S(\theta_S) = U_S(\underline{\theta}_S) - \int_{\underline{\theta}_S}^{\theta_S} \overline{p}_S(z)\,dz, \quad U_B(\theta_B) = U_B(\underline{\theta}_B) + \int_{\underline{\theta}_B}^{\theta_B} \overline{p}_B(z)\,dz.$$

Interim individual rationality: For $i = S, B$, $U_i(\theta_i) \geq \hat{u}_i(\theta_i) = 0$ for all θ_i.

Incentive compatibility and individual rationality

Let $\overline{p}_S(\theta_S) := E_{\theta_B}[p(\theta_S, \theta_B)]$ and $\overline{p}_B(\theta_B) := E_{\theta_S}[p(\theta_S, \theta_B)]$.
Hence, $U_S(\theta_S) = \bar{t}_S(\theta_S) - \overline{p}_S(\theta_S)\theta_S$ and $U_B(\theta_B) = \overline{p}_B(\theta_B)\theta_B + \bar{t}_B(\theta_B)$.

Bayesian incentive compatibility is equivalent to (IC1) and (IC2).

1. (IC1): $\overline{p}_S(\theta_S)$ is *non-increasing* and $\overline{p}_B(\theta_B)$ is *non-decreasing*.
2. (IC2):

$$U_S(\theta_S) = U_S(\underline{\theta}_S) - \int_{\underline{\theta}_S}^{\theta_S} \overline{p}_S(z)\,dz, \quad U_B(\theta_B) = U_B(\underline{\theta}_B) + \int_{\underline{\theta}_B}^{\theta_B} \overline{p}_B(z)\,dz.$$

Interim individual rationality: For $i = S, B$, $U_i(\theta_i) \geq \hat{u}_i(\theta_i) = 0$ for all θ_i.

- (IC1) and (IC2) imply $U'_S(\theta_S) \leq 0$ and $U'_B(\theta_B) \geq 0$.
- Incentive compatible mechanisms are interim IR if

$$U_S(\overline{\theta}_S) \geq 0 \quad \text{and} \quad U_B(\underline{\theta}_B) \geq 0.$$

5.2 The Myerson–Satterthwaite Theorem

Proposition (Myerson–Satterthwaite Theorem)

For the environment described in 5.1, there is no Bayesian incentive compatible direct mechanism with ex post efficient SCF that is interim individually rational for the seller and the buyer.

Proof

Consider the VCG mechanism with

$$h_S(\theta_B) = \min\{0, \overline{\theta}_S - \theta_B\} \quad \text{and} \quad h_B(\theta_S) = -\max\{0, \underline{\theta}_B - \theta_S\}.$$

Proof

Consider the VCG mechanism with

$$h_S(\theta_B) = \min\{0, \overline{\theta}_S - \theta_B\} \quad \text{and} \quad h_B(\theta_S) = -\max\{0, \underline{\theta}_B - \theta_S\}.$$

This dominant strategy implements allocation rule $p^*(\theta_S, \theta_B)$ with payment rule

$$t_S(\theta_S, \theta_B) = \begin{cases} \min\{\theta_B, \overline{\theta}_S\} & \text{if } \theta_S < \theta_B, \\ 0 & \text{if } \theta_S > \theta_B, \end{cases}$$

$$t_B(\theta_S, \theta_B) = \begin{cases} -\max\{\theta_S, \underline{\theta}_B\} & \text{if } \theta_S < \theta_B, \\ 0 & \text{if } \theta_S > \theta_B. \end{cases}$$

Proof

Consider the VCG mechanism with

$$h_S(\theta_B) = \min\{0, \overline{\theta}_S - \theta_B\} \quad \text{and} \quad h_B(\theta_S) = -\max\{0, \underline{\theta}_B - \theta_S\}.$$

This dominant strategy implements allocation rule $p^*(\theta_S, \theta_B)$ with payment rule

$$t_S(\theta_S, \theta_B) = \begin{cases} \min\{\theta_B, \overline{\theta}_S\} & \text{if } \theta_S < \theta_B, \\ 0 & \text{if } \theta_S > \theta_B, \end{cases}$$

$$t_B(\theta_S, \theta_B) = \begin{cases} -\max\{\theta_S, \underline{\theta}_B\} & \text{if } \theta_S < \theta_B, \\ 0 & \text{if } \theta_S > \theta_B. \end{cases}$$

$\Longrightarrow$ $U_S(\overline{\theta}_S) = 0$ and $U_B(\underline{\theta}_B) = 0$.

$\Longrightarrow$ Interim individual rationality is satisfied.

Proof

Consider the VCG mechanism with

$$h_S(\theta_B) = \min\{0, \overline{\theta}_S - \theta_B\} \quad \text{and} \quad h_B(\theta_S) = -\max\{0, \underline{\theta}_B - \theta_S\}.$$

This dominant strategy implements allocation rule $p^*(\theta_S, \theta_B)$ with payment rule

$$t_S(\theta_S, \theta_B) = \begin{cases} \min\{\theta_B, \overline{\theta}_S\} & \text{if } \theta_S < \theta_B, \\ 0 & \text{if } \theta_S > \theta_B, \end{cases}$$

$$t_B(\theta_S, \theta_B) = \begin{cases} -\max\{\theta_S, \underline{\theta}_B\} & \text{if } \theta_S < \theta_B, \\ 0 & \text{if } \theta_S > \theta_B. \end{cases}$$

$\Longrightarrow$ $U_S(\overline{\theta}_S) = 0$ and $U_B(\underline{\theta}_B) = 0$.

$\Longrightarrow$ Interim individual rationality is satisfied.

However, budget balance is not satisfied: If $\theta_S < \theta_B$,

$$t_S(\theta_S, \theta_B) + t_B(\theta_S, \theta_B) = \min\{\theta_B, \overline{\theta}_S\} - \max\{\theta_S, \underline{\theta}_B\} > 0.$$

Let $\tilde{U}_S(\theta_S)$ and $\tilde{U}_B(\theta_B)$ denote the interim utilities in the VCG mechanism.

- Because of (IC2), the interim utilities in every Bayesian incentive compatible direct mechanism with allocation rule p^* satisfy

$$U_i(\theta_i) = \tilde{U}_i(\theta_i) + D_i \qquad \text{for some constants } D_S \text{ and } D_B.$$

Let $\tilde{U}_S(\theta_S)$ and $\tilde{U}_B(\theta_B)$ denote the interim utilities in the VCG mechanism.

- Because of (IC2), the interim utilities in every Bayesian incentive compatible direct mechanism with allocation rule p^* satisfy

$$U_i(\theta_i) = \tilde{U}_i(\theta_i) + D_i \qquad \text{for some constants } D_S \text{ and } D_B.$$

  Since $\tilde{U}_S(\overline{\theta}_S) = \tilde{U}_B(\underline{\theta}_B) = 0$, interim IR requires $D_S \geq 0$ and $D_B \geq 0$.

Let $\tilde{U}_S(\theta_S)$ and $\tilde{U}_B(\theta_B)$ denote the interim utilities in the VCG mechanism.

- Because of (IC2), the interim utilities in every Bayesian incentive compatible direct mechanism with allocation rule p^* satisfy

  $$U_i(\theta_i) = \tilde{U}_i(\theta_i) + D_i \qquad \text{for some constants } D_S \text{ and } D_B.$$

  Since $\tilde{U}_S(\overline{\theta}_S) = \tilde{U}_B(\underline{\theta}_B) = 0$, interim IR requires $D_S \geq 0$ and $D_B \geq 0$.

- Budget balance implies $E[t_S(\theta_S, \theta_B) + t_B(\theta_S, \theta_B)] = 0$.

Let $\tilde{U}_S(\theta_S)$ and $\tilde{U}_B(\theta_B)$ denote the interim utilities in the VCG mechanism.

- Because of (IC2), the interim utilities in every Bayesian incentive compatible direct mechanism with allocation rule p^* satisfy

$$U_i(\theta_i) = \tilde{U}_i(\theta_i) + D_i \qquad \text{for some constants } D_S \text{ and } D_B.$$

  Since $\tilde{U}_S(\overline{\theta}_S) = \tilde{U}_B(\underline{\theta}_B) = 0$, interim IR requires $D_S \geq 0$ and $D_B \geq 0$.

- Budget balance implies $E[t_S(\theta_S, \theta_B) + t_B(\theta_S, \theta_B)] = 0$.

- Note that $\bar{t}_i(\theta_i) = E_{\theta_{-i}}[\tilde{t}_i(\theta_S, \theta_B)] + D_i$ for $i = S, B$. For every incentive compatible and individually rational mechanism that implements p^*,

$$\begin{aligned} E[t_S(\theta_S, \theta_B) + t_B(\theta_S, \theta_B)] &= E[\bar{t}_S(\theta_S) + \bar{t}_B(\theta_B)] \\ &= E[\tilde{t}_S(\theta_S, \theta_B) + \tilde{t}_B(\theta_S, \theta_B)] + D_S + D_B > 0. \end{aligned}$$

Let $\tilde{U}_S(\theta_S)$ and $\tilde{U}_B(\theta_B)$ denote the interim utilities in the VCG mechanism.

- Because of (IC2), the interim utilities in every Bayesian incentive compatible direct mechanism with allocation rule p^* satisfy

$$U_i(\theta_i) = \tilde{U}_i(\theta_i) + D_i \qquad \text{for some constants } D_S \text{ and } D_B.$$

  Since $\tilde{U}_S(\overline{\theta}_S) = \tilde{U}_B(\underline{\theta}_B) = 0$, interim IR requires $D_S \geq 0$ and $D_B \geq 0$.

- Budget balance implies $E[t_S(\theta_S, \theta_B) + t_B(\theta_S, \theta_B)] = 0$.

- Note that $\bar{t}_i(\theta_i) = E_{\theta_{-i}}[\tilde{t}_i(\theta_S, \theta_B)] + D_i$ for $i = S, B$. For every incentive compatible and individually rational mechanism that implements p^*,

$$\begin{aligned} E[t_S(\theta_S, \theta_B) + t_B(\theta_S, \theta_B)] &= E[\bar{t}_S(\theta_S) + \bar{t}_B(\theta_B)] \\ &= E[\tilde{t}_S(\theta_S, \theta_B) + \tilde{t}_B(\theta_S, \theta_B)] + D_S + D_B > 0. \end{aligned}$$

  Hence, no Bayesian incentive compatible and interim individually rational direct mechanism with SCF (p^*, t) satisfies budget balance. □

Discussion

- The proof reveals that the impossibility result still holds if we only require the budget to balance *in expectation* instead of ex post.

Discussion

- The proof reveals that the impossibility result still holds if we only require the budget to balance *in expectation* instead of ex post.

- According to the *Coase Theorem*, under complete information and if there are no transactions costs, the initial allocation of property rights is irrelevant for an ex post efficient outcome, because agents will continue to engage in transactions as long as the allocation remains inefficient.

Discussion

- The proof reveals that the impossibility result still holds if we only require the budget to balance *in expectation* instead of ex post.

- According to the *Coase Theorem*, under complete information and if there are no transactions costs, the initial allocation of property rights is irrelevant for an ex post efficient outcome, because agents will continue to engage in transactions as long as the allocation remains inefficient.

- The Myerson-Satterthwaite Theorem shows that this is not valid under private information:
  There is no voluntary trading institution or bargaining process that ensures ex post efficient reallocation of the seller's property right to the object.

If we depart from the assumptions in 5.1, the impossibility result need not hold.

- Suppose the intervals $[\underline{\theta}_S, \overline{\theta}_S]$ and $[\underline{\theta}_B, \overline{\theta}_B]$ do not overlap:
  - If $\overline{\theta}_B < \underline{\theta}_S$, no trade/payments is ex post efficient and implementable.
  - If $\overline{\theta}_S < \underline{\theta}_B$, it is efficient to always trade. $p^*(\theta_S, \theta_B) = 1$ can be implemented with a fixed price $t_S(\theta_S, \theta_B) = -t_B(\theta_S, \theta_B) = x \in [\overline{\theta}_S, \underline{\theta}_B]$.

If we depart from the assumptions in 5.1, the impossibility result need not hold.

- Suppose the intervals $[\underline{\theta}_S, \overline{\theta}_S]$ and $[\underline{\theta}_B, \overline{\theta}_B]$ do not overlap:
  - If $\overline{\theta}_B < \underline{\theta}_S$, no trade/payments is ex post efficient and implementable.
  - If $\overline{\theta}_S < \underline{\theta}_B$, it is efficient to always trade. $p^*(\theta_S, \theta_B) = 1$ can be implemented with a fixed price $t_S(\theta_S, \theta_B) = -t_B(\theta_S, \theta_B) = x \in [\overline{\theta}_S, \underline{\theta}_B]$.

- Suppose types are drawn from discrete distributions:
  - *In the tutorial (23.E.3), you will be asked to find an example where ex post efficient trade is IC and interim IR.*

If we depart from the assumptions in 5.1, the impossibility result need not hold.

- Suppose the intervals $[\underline{\theta}_S, \overline{\theta}_S]$ and $[\underline{\theta}_B, \overline{\theta}_B]$ do not overlap:
  - If $\overline{\theta}_B < \underline{\theta}_S$, no trade/payments is ex post efficient and implementable.
  - If $\overline{\theta}_S < \underline{\theta}_B$, it is efficient to always trade. $p^*(\theta_S, \theta_B) = 1$ can be implemented with a fixed price $t_S(\theta_S, \theta_B) = -t_B(\theta_S, \theta_B) = x \in [\overline{\theta}_S, \underline{\theta}_B]$.

- Suppose types are drawn from discrete distributions:
  - *In the tutorial (23.E.3), you will be asked to find an example where ex post efficient trade is IC and interim IR.*

- Suppose we require only ex ante IR instead of interim IR:
  - The *expected externality mechanism* is ex post efficient and Bayesian IC.
  - With an additional fixed payment from the buyer to the seller we can transfer ex ante expected utility between agents and achieve ex ante IR.

6 Optimal Auctions

(Myerson, 1981)

6.1 Preliminaries

Consider a seller with a single object who faces several potential buyers.

The seller has many options when choosing the selling procedure:

- posted price
- (non-)standard auction
- negotiate with one or several buyers
- ...

6.1 Preliminaries

Consider a seller with a single object who faces several potential buyers.

The seller has many options when choosing the selling procedure:

- posted price
- (non-)standard auction
- negotiate with one or several buyers
- ...

We are now in a position to answer the question from Section 1:

- What kind of mechanisms are *optimal* (revenue maximizing) for the seller?
- Put differently: Which implementable SCFs maximize the seller's utility?

Environment

- Single indivisible object.
- Agent 0: seller with valuation $\theta_0 = 0$ (no private information).
- Agent $i \in N := \{1, 2, \ldots, n\}$: buyer with valuation $\theta_i \in [\underline{\theta}_i, \overline{\theta}_i]$, $\overline{\theta}_i > 0$.
- Each θ_i is independently drawn from a continuous distribution F_i with density $f_i(\theta_i) > 0$ for all $\theta_i \in [\underline{\theta}_i, \overline{\theta}_i]$.

Environment

- Single indivisible object.
- Agent 0: seller with valuation $\theta_0 = 0$ (no private information).
- Agent $i \in N := \{1, 2, \ldots, n\}$: buyer with valuation $\theta_i \in [\underline{\theta}_i, \overline{\theta}_i]$, $\overline{\theta}_i > 0$.
- Each θ_i is independently drawn from a continuous distribution F_i with density $f_i(\theta_i) > 0$ for all $\theta_i \in [\underline{\theta}_i, \overline{\theta}_i]$.
- Set of possible allocations:

$$K = \Big\{(q_1, \ldots, q_n) \in [0,1]^n \;\Big|\; \textstyle\sum_{i \in N} q_i \leq 1\Big\}.$$

  - Buyer i obtains the object with probability q_i.
  - With probability $q_0 := 1 - \sum_{i \in N} q_i$ the seller keeps the object.

Environment

- Single indivisible object.
- Agent 0: seller with valuation $\theta_0 = 0$ (no private information).
- Agent $i \in N := \{1, 2, \ldots, n\}$: buyer with valuation $\theta_i \in [\underline{\theta}_i, \overline{\theta}_i]$, $\overline{\theta}_i > 0$.
- Each θ_i is independently drawn from a continuous distribution F_i with density $f_i(\theta_i) > 0$ for all $\theta_i \in [\underline{\theta}_i, \overline{\theta}_i]$.
- Set of possible allocations:

$$K = \left\{ (q_1, \ldots, q_n) \in [0,1]^n \;\middle|\; \sum_{i \in N} q_i \leq 1 \right\}.$$

  - Buyer i obtains the object with probability q_i.
  - With probability $q_0 := 1 - \sum_{i \in N} q_i$ the seller keeps the object.

- Utility functions: $u_i((k,t), \theta_i) = \theta_i q_i + t_i$ (i.e., linear with $v_i(k) = q_i$).

Restriction to **budget balanced transfers**: $t_0 = -\sum_{i \in N} t_i$.

- Without loss of generality: since the seller has no private information, if alternative with $t_0 < -\sum_{i \in N} t_i$ is implementable, then $\tilde{t}_0 = -\sum_{i \in N} t_i$ is implementable, making the seller strictly better off.

Restriction to **budget balanced transfers**: $t_0 = -\sum_{i \in N} t_i$.

- Without loss of generality: since the seller has no private information, if alternative with $t_0 < -\sum_{i \in N} t_i$ is implementable, then $\tilde{t}_0 = -\sum_{i \in N} t_i$ is implementable, making the seller strictly better off.

Outside option: $\hat{u}_i(\theta_i) = 0$ for all i and $\theta_i \in [\underline{\theta}_i, \overline{\theta}_i]$.

We require **interim individual rationality**.

- A bidder in an auction mechanism decides whether to participate while knowing his valuation. By submitting a bid, he commits to accept the auction outcome.

Restriction to **budget balanced transfers**: $t_0 = -\sum_{i \in N} t_i$.

- Without loss of generality: since the seller has no private information, if alternative with $t_0 < -\sum_{i \in N} t_i$ is implementable, then $\tilde{t}_0 = -\sum_{i \in N} t_i$ is implementable, making the seller strictly better off.

Outside option: $\hat{u}_i(\theta_i) = 0$ for all i and $\theta_i \in [\underline{\theta}_i, \overline{\theta}_i]$.

We require **interim individual rationality**.

- A bidder in an auction mechanism decides whether to participate while knowing his valuation. By submitting a bid, he commits to accept the auction outcome.

Social choice function: For all $\theta \in [\underline{\theta}_1, \overline{\theta}_1] \times \cdots \times [\underline{\theta}_n, \overline{\theta}_n]$,

- allocation rule $q(\theta) := (q_1(\theta), \ldots, q_n(\theta)) \in K$,
- payment rule $t(\theta) := (t_1(\theta), \ldots, t_n(\theta)) \in \mathbb{R}^n$

  (and by budget balance $t_0(\theta) = -\sum_{i \in N} t_i(\theta)$).

Maximizing the seller's expected revenue

Revelation principle: restrict attention to Bayesian incentive compatible direct mechanisms with SCF (q, t).

Maximizing the seller's expected revenue

Revelation principle: restrict attention to Bayesian incentive compatible direct mechanisms with SCF (q, t).

Seller's expected revenue: $E[t_0(\theta)] = E[-\sum_{i \in N} t_i(\theta)]$.

Maximizing the seller's expected revenue

Revelation principle: restrict attention to Bayesian incentive compatible direct mechanisms with SCF (q, t).

Seller's expected revenue: $E[t_0(\theta)] = E[-\sum_{i \in N} t_i(\theta)]$.

Constrained optimization problem:

$$\max_{(q,t)} E\Big[-\sum_{i \in N} t_i(\theta)\Big]$$

subject to

- Bayesian incentive compatibility
- interim individual rationality

Incentive compatibility and individual rationality

Recall the notation from Section 4:

- $\overline{q}_i(\theta_i) := \overline{v}_i(\theta_i) = E_{\theta_{-i}}[q_i(\theta_i, \theta_{-i})]$ and $\overline{t}_i(\theta_i) = E_{\theta_{-i}}[t_i(\theta_i, \theta_{-i})]$,
- $U_i(\theta_i) = \theta_i \overline{q}_i(\theta_i) + \overline{t}_i(\theta_i)$.

Incentive compatibility and individual rationality

Recall the notation from Section 4:

- $\overline{q}_i(\theta_i) := \overline{v}_i(\theta_i) = E_{\theta_{-i}}[q_i(\theta_i, \theta_{-i})]$ and $\overline{t}_i(\theta_i) = E_{\theta_{-i}}[t_i(\theta_i, \theta_{-i})]$,
- $U_i(\theta_i) = \theta_i \overline{q}_i(\theta_i) + \overline{t}_i(\theta_i)$.

Bayesian incentive compatibility is equivalent to (IC1) and (IC2), i.e.,

- $\overline{q}_i(\theta_i)$ is non-decreasing,
- $U_i(\theta_i) = U_i(\underline{\theta}_i) + \int_{\underline{\theta}_i}^{\theta_i} \overline{q}_i(z)\, dz.$

$$\iff \quad -\overline{t}_i(\theta_i) = -U_i(\underline{\theta}_i) + \theta_i \overline{q}_i(\theta_i) - \int_{\underline{\theta}_i}^{\theta_i} \overline{q}_i(z)\, dz.$$

Incentive compatibility and individual rationality

Recall the notation from Section 4:

- $\overline{q}_i(\theta_i) := \overline{v}_i(\theta_i) = E_{\theta_{-i}}[q_i(\theta_i, \theta_{-i})]$ and $\overline{t}_i(\theta_i) = E_{\theta_{-i}}[t_i(\theta_i, \theta_{-i})]$,
- $U_i(\theta_i) = \theta_i \overline{q}_i(\theta_i) + \overline{t}_i(\theta_i)$.

Bayesian incentive compatibility is equivalent to (IC1) and (IC2), i.e.,

- $\overline{q}_i(\theta_i)$ is non-decreasing,
- $U_i(\theta_i) = U_i(\underline{\theta}_i) + \int_{\underline{\theta}_i}^{\theta_i} \overline{q}_i(z)\, dz.$

$$\iff \quad -\overline{t}_i(\theta_i) = -U_i(\underline{\theta}_i) + \theta_i \overline{q}_i(\theta_i) - \int_{\underline{\theta}_i}^{\theta_i} \overline{q}_i(z)\, dz.$$

Interim individual rationality requires $U_i(\theta_i) \geq 0$ for all $\theta_i \in [\underline{\theta}_i, \overline{\theta}_i]$.

Incentive compatibility and individual rationality

Recall the notation from Section 4:

- $\overline{q}_i(\theta_i) := \overline{v}_i(\theta_i) = E_{\theta_{-i}}[q_i(\theta_i, \theta_{-i})]$ and $\overline{t}_i(\theta_i) = E_{\theta_{-i}}[t_i(\theta_i, \theta_{-i})]$,
- $U_i(\theta_i) = \theta_i \overline{q}_i(\theta_i) + \overline{t}_i(\theta_i)$.

Bayesian incentive compatibility is equivalent to (IC1) and (IC2), i.e.,

- $\overline{q}_i(\theta_i)$ is non-decreasing,
- $U_i(\theta_i) = U_i(\underline{\theta}_i) + \int_{\underline{\theta}_i}^{\theta_i} \overline{q}_i(z)\, dz.$

$$\iff \quad -\overline{t}_i(\theta_i) = -U_i(\underline{\theta}_i) + \theta_i \overline{q}_i(\theta_i) - \int_{\underline{\theta}_i}^{\theta_i} \overline{q}_i(z)\, dz.$$

Interim individual rationality requires $U_i(\theta_i) \geq 0$ for all $\theta_i \in [\underline{\theta}_i, \overline{\theta}_i]$.

(IC1) and (IC2) imply $U_i(\theta_i) \geq U_i(\underline{\theta}_i)$ for all θ_i.

$\Rightarrow$ Incentive compatible mechanisms are individually rational if $U_i(\underline{\theta}_i) \geq 0$.

The optimization problem

Note that $E[-\sum_{i\in N} t_i(\theta)] = \sum_{i\in N} E[-t_i(\theta)] = \sum_{i\in N} E[-\bar{t}_i(\theta_i)]$.

The optimization problem

Note that $E[-\sum_{i\in N} t_i(\theta)] = \sum_{i\in N} E[-t_i(\theta)] = \sum_{i\in N} E[-\bar{t}_i(\theta_i)]$.

The constrained optimization problem can be written as

$$\max_{(q,t)} \sum_{i\in N} E[-\bar{t}_i(\theta_i)]$$

subject to

1. $\bar{q}_i(\theta_i)$ is non-decreasing,
2. $-\bar{t}_i(\theta_i) = -U_i(\underline{\theta}_i) + \theta_i \bar{q}_i(\theta_i) - \int_{\underline{\theta}_i}^{\theta_i} \bar{q}_i(z)\,dz$,
3. $U_i(\underline{\theta}_i) \geq 0$.

The optimization problem

Note that $E[-\sum_{i\in N} t_i(\theta)] = \sum_{i\in N} E[-t_i(\theta)] = \sum_{i\in N} E[-\bar{t}_i(\theta_i)]$.

The constrained optimization problem can be written as

$$\max_{(q,t)} \sum_{i\in N} E[\,-\,\bar{t}_i(\theta_i)]$$

subject to

1. $\bar{q}_i(\theta_i)$ is non-decreasing,
2. $-\bar{t}_i(\theta_i) = -\,U_i(\underline{\theta}_i) + \theta_i \bar{q}_i(\theta_i) - \int_{\underline{\theta}_i}^{\theta_i} \bar{q}_i(z)\,dz$,
3. $U_i(\underline{\theta}_i) \geq 0$.

$\rightarrow$ We use the 2nd constraint to substitute for $-\bar{t}_i(\theta_i)$ in the objective function.

Ex ante expected payments

$$E[-\bar{t}_i(\theta_i)] = -\int_{\underline{\theta}_i}^{\overline{\theta}_i} \bar{t}_i(\theta_i) f_i(\theta_i)\, d\theta_i$$

Ex ante expected payments

$$
\begin{aligned}
E[-\bar{t}_i(\theta_i)] &= -\int_{\underline{\theta}_i}^{\overline{\theta}_i} \bar{t}_i(\theta_i) f_i(\theta_i)\, d\theta_i \\
&= -U_i(\underline{\theta}_i) + \int_{\underline{\theta}_i}^{\overline{\theta}_i} \theta_i \bar{q}_i(\theta_i) f_i(\theta_i)\, d\theta_i - \int_{\underline{\theta}_i}^{\overline{\theta}_i} \int_{\underline{\theta}_i}^{\theta_i} \bar{q}_i(z)\, dz\, f_i(\theta_i)\, d\theta_i.
\end{aligned}
$$

Ex ante expected payments

$$
\begin{aligned}
E[-\overline{t}_i(\theta_i)] &= -\int_{\underline{\theta}_i}^{\overline{\theta}_i} \overline{t}_i(\theta_i) f_i(\theta_i)\, d\theta_i \\
&= -U_i(\underline{\theta}_i) + \int_{\underline{\theta}_i}^{\overline{\theta}_i} \theta_i \overline{q}_i(\theta_i) f_i(\theta_i)\, d\theta_i - \int_{\underline{\theta}_i}^{\overline{\theta}_i} \int_{\underline{\theta}_i}^{\theta_i} \overline{q}_i(z)\, dz\, f_i(\theta_i)\, d\theta_i.
\end{aligned}
$$

Interchanging the order of integration:

$$
\int_{\underline{\theta}_i}^{\overline{\theta}_i} \int_{\underline{\theta}_i}^{\theta_i} \overline{q}_i(z)\, dz\, f_i(\theta_i)\, d\theta_i = \int_{\underline{\theta}_i}^{\overline{\theta}_i} \int_{z}^{\overline{\theta}_i} \overline{q}_i(z)\; f_i(\theta_i)\, d\theta_i dz
$$

Ex ante expected payments

$$
\begin{aligned}
E[-\overline{t}_i(\theta_i)] &= -\int_{\underline{\theta}_i}^{\overline{\theta}_i} \overline{t}_i(\theta_i) f_i(\theta_i)\, d\theta_i \\
&= -U_i(\underline{\theta}_i) + \int_{\underline{\theta}_i}^{\overline{\theta}_i} \theta_i \overline{q}_i(\theta_i) f_i(\theta_i)\, d\theta_i - \int_{\underline{\theta}_i}^{\overline{\theta}_i} \int_{\underline{\theta}_i}^{\theta_i} \overline{q}_i(z)\, dz\, f_i(\theta_i)\, d\theta_i.
\end{aligned}
$$

Interchanging the order of integration:

$$
\int_{\underline{\theta}_i}^{\overline{\theta}_i} \int_{\underline{\theta}_i}^{\theta_i} \overline{q}_i(z)\, dz\, f_i(\theta_i)\, d\theta_i = \int_{\underline{\theta}_i}^{\overline{\theta}_i} \int_{z}^{\overline{\theta}_i} \overline{q}_i(z)\; f_i(\theta_i)\, d\theta_i dz = \int_{\underline{\theta}_i}^{\overline{\theta}_i} \overline{q}_i(\theta_i)\big(1 - F_i(\theta_i)\big)\, d\theta_i.
$$

Ex ante expected payments

$$
\begin{aligned}
E[-\bar{t}_i(\theta_i)] &= -\int_{\underline{\theta}_i}^{\overline{\theta}_i} \bar{t}_i(\theta_i) f_i(\theta_i)\, d\theta_i \\
&= -U_i(\underline{\theta}_i) + \int_{\underline{\theta}_i}^{\overline{\theta}_i} \theta_i \overline{q}_i(\theta_i) f_i(\theta_i)\, d\theta_i - \int_{\underline{\theta}_i}^{\overline{\theta}_i} \int_{\underline{\theta}_i}^{\theta_i} \overline{q}_i(z)\, dz\, f_i(\theta_i)\, d\theta_i.
\end{aligned}
$$

Interchanging the order of integration:

$$
\int_{\underline{\theta}_i}^{\overline{\theta}_i} \int_{\underline{\theta}_i}^{\theta_i} \overline{q}_i(z)\, dz\, f_i(\theta_i)\, d\theta_i = \int_{\underline{\theta}_i}^{\overline{\theta}_i} \int_{z}^{\overline{\theta}_i} \overline{q}_i(z)\; f_i(\theta_i)\, d\theta_i dz = \int_{\underline{\theta}_i}^{\overline{\theta}_i} \overline{q}_i(\theta_i) \big(1 - F_i(\theta_i)\big)\, d\theta_i.
$$

$$
E[-\bar{t}_i(\theta_i)] = -U_i(\underline{\theta}_i) + \int_{\underline{\theta}_i}^{\overline{\theta}_i} \overline{q}_i(\theta_i) \left(\theta_i - \frac{1 - F_i(\theta_i)}{f_i(\theta_i)} \right) f_i(\theta_i)\, d\theta_i
$$

Ex ante expected payments

$$
\begin{aligned}
E[-\bar{t}_i(\theta_i)] &= -\int_{\underline{\theta}_i}^{\overline{\theta}_i} \bar{t}_i(\theta_i) f_i(\theta_i)\, d\theta_i \\
&= -U_i(\underline{\theta}_i) + \int_{\underline{\theta}_i}^{\overline{\theta}_i} \theta_i \overline{q}_i(\theta_i) f_i(\theta_i)\, d\theta_i - \int_{\underline{\theta}_i}^{\overline{\theta}_i} \int_{\underline{\theta}_i}^{\theta_i} \overline{q}_i(z)\, dz\, f_i(\theta_i)\, d\theta_i.
\end{aligned}
$$

Interchanging the order of integration:

$$
\int_{\underline{\theta}_i}^{\overline{\theta}_i} \int_{\underline{\theta}_i}^{\theta_i} \overline{q}_i(z)\, dz\, f_i(\theta_i)\, d\theta_i = \int_{\underline{\theta}_i}^{\overline{\theta}_i} \int_{z}^{\overline{\theta}_i} \overline{q}_i(z)\; f_i(\theta_i)\, d\theta_i dz = \int_{\underline{\theta}_i}^{\overline{\theta}_i} \overline{q}_i(\theta_i)\big(1 - F_i(\theta_i)\big)\, d\theta_i.
$$

$$
\begin{aligned}
E[-\bar{t}_i(\theta_i)] &= -U_i(\underline{\theta}_i) + \int_{\underline{\theta}_i}^{\overline{\theta}_i} \overline{q}_i(\theta_i) \left(\theta_i - \frac{1 - F_i(\theta_i)}{f_i(\theta_i)}\right) f_i(\theta_i)\, d\theta_i \\
&= -U_i(\underline{\theta}_i) + E_{\theta_i}\left[\overline{q}_i(\theta_i) \left(\theta_i - \frac{1 - F_i(\theta_i)}{f_i(\theta_i)}\right)\right].
\end{aligned}
$$

Define

$$\psi_i(\theta_i) := \theta_i - \frac{1 - F_i(\theta_i)}{f_i(\theta_i)} \qquad \text{(buyer } i\text{'s } \textit{virtual valuation}\text{).}$$

Expected revenue:

Define

$$\psi_i(\theta_i) := \theta_i - \frac{1 - F_i(\theta_i)}{f_i(\theta_i)} \qquad \text{(buyer } i\text{'s } \textit{virtual valuation}\text{).}$$

Expected revenue:

$$\sum_{i \in N} E[-\bar{t}_i(\theta_i)] = \sum_{i \in N} E_{\theta_i}[\bar{q}_i(\theta_i)\psi_i(\theta_i)] - \sum_{i \in N} U_i(\underline{\theta}_i)$$

Define

$$\psi_i(\theta_i) := \theta_i - \frac{1 - F_i(\theta_i)}{f_i(\theta_i)} \qquad \text{(buyer } i\text{'s } \textit{virtual valuation}\text{)}.$$

Expected revenue:

$$\begin{aligned}\sum_{i \in N} E[-\bar{t}_i(\theta_i)] &= \sum_{i \in N} E_{\theta_i}[\bar{q}_i(\theta_i)\psi_i(\theta_i)] - \sum_{i \in N} U_i(\underline{\theta}_i) \\ &= \sum_{i \in N} E_{\theta}[q_i(\theta)\psi_i(\theta_i)] - \sum_{i \in N} U_i(\underline{\theta}_i)\end{aligned}$$

Define

$$\psi_i(\theta_i) := \theta_i - \frac{1 - F_i(\theta_i)}{f_i(\theta_i)} \qquad \text{(buyer } i\text{'s } \textit{virtual valuation}\text{).}$$

Expected revenue:

$$\begin{aligned}
\sum_{i\in N} E[-\bar{t}_i(\theta_i)] &= \sum_{i\in N} E_{\theta_i}[\bar{q}_i(\theta_i)\psi_i(\theta_i)] - \sum_{i\in N} U_i(\underline{\theta}_i) \\
&= \sum_{i\in N} E_{\theta}[q_i(\theta)\psi_i(\theta_i)] - \sum_{i\in N} U_i(\underline{\theta}_i) \\
&= E_{\theta}\left[\sum_{i\in N} q_i(\theta)\psi_i(\theta_i)\right] - \sum_{i\in N} U_i(\underline{\theta}_i)
\end{aligned}$$

6.2 Revenue Maximizing Mechanisms

Revenue maximizing direct mechanisms (q, t) solve

$$\max_{\substack{q,\\ \bar{t}_1(\underline{\theta}_1),\ldots,\bar{t}_n(\underline{\theta}_n)}} E_\theta\left[\sum_{i\in N} q_i(\theta)\psi_i(\theta_i)\right] - \sum_{i\in N} U_i(\underline{\theta}_i)$$

s.t.

1. $\overline{q}_i(\theta_i)$ is non-decreasing,
2. $U_i(\underline{\theta}_i) \geq 0.$

6.2 Revenue Maximizing Mechanisms

Revenue maximizing direct mechanisms (q, t) solve

$$\max_{\substack{q,\\ \bar{t}_1(\underline{\theta}_1),\ldots,\bar{t}_n(\underline{\theta}_n)}} E_\theta\left[\sum_{i\in N} q_i(\theta)\psi_i(\theta_i)\right] - \sum_{i\in N} U_i(\underline{\theta}_i)$$

s.t.

1. $\overline{q}_i(\theta_i)$ is non-decreasing,
2. $U_i(\underline{\theta}_i) \geq 0$.

- The 2nd constraint is binding: optimal payment rules t are such that

$$U_i(\underline{\theta}_i) = 0 \quad \Longleftrightarrow \quad -\bar{t}_i(\underline{\theta}_i) = \underline{\theta}_i \overline{q}_i(\underline{\theta}_i).$$

6.2 Revenue Maximizing Mechanisms

Revenue maximizing direct mechanisms (q, t) solve

$$\max_{\substack{q,\\ \bar{t}_1(\underline{\theta}_1),\dots,\bar{t}_n(\underline{\theta}_n)}} \quad E_\theta\left[\sum_{i\in N} q_i(\theta)\psi_i(\theta_i)\right] - \sum_{i\in N} U_i(\underline{\theta}_i)$$

s.t.

1. $\bar{q}_i(\theta_i)$ is non-decreasing,
2. $U_i(\underline{\theta}_i) \geq 0.$

- The 2nd constraint is binding: optimal payment rules t are such that

$$U_i(\underline{\theta}_i) = 0 \quad \Longleftrightarrow \quad -\bar{t}_i(\underline{\theta}_i) = \underline{\theta}_i\bar{q}_i(\underline{\theta}_i).$$

- The problem reduces to choosing the allocation rule q to maximize the first part of the objective subject to $\bar{q}_i(\theta_i)$ being non-decreasing.

Regularity and pointwise maximum

Regularity assumption: For all $i \in N$, the distribution F_i is such that

$$\psi_i(\theta_i) \text{ is strictly increasing in } \theta_i. \tag{R}$$

Regularity and pointwise maximum

Regularity assumption: For all $i \in N$, the distribution F_i is such that

$$\psi_i(\theta_i) \text{ is strictly increasing in } \theta_i. \tag{R}$$

- Ignoring the monotonicity constraint on $\overline{q}_i(\theta_i)$, the objective function is maximized if q maximizes $\sum_{i \in N} q_i(\theta)\psi_i(\theta_i)$ for every possible θ,

Regularity and pointwise maximum

Regularity assumption: For all $i \in N$, the distribution F_i is such that

$$\psi_i(\theta_i) \text{ is strictly increasing in } \theta_i. \qquad \text{(R)}$$

- Ignoring the monotonicity constraint on $\overline{q}_i(\theta_i)$, the objective function is maximized if q maximizes $\sum_{i \in N} q_i(\theta)\psi_i(\theta_i)$ for every possible θ, i.e.,

$$q_i(\theta) = \begin{cases} 1 & \text{if } \psi_i(\theta_i) > \max\{0, \max_{j \neq i} \psi_j(\theta_j)\}, \\ 0 & \text{if } \psi_i(\theta_i) < \max\{0, \max_{j \neq i} \psi_j(\theta_j)\}. \end{cases}$$

Regularity and pointwise maximum

Regularity assumption: For all $i \in N$, the distribution F_i is such that

$$\psi_i(\theta_i) \text{ is strictly increasing in } \theta_i. \tag{R}$$

- Ignoring the monotonicity constraint on $\overline{q}_i(\theta_i)$, the objective function is maximized if q maximizes $\sum_{i \in N} q_i(\theta)\psi_i(\theta_i)$ for every possible θ, i.e.,

$$q_i(\theta) = \begin{cases} 1 & \text{if } \psi_i(\theta_i) > \max\{0, \max_{j \neq i} \psi_j(\theta_j)\}, \\ 0 & \text{if } \psi_i(\theta_i) < \max\{0, \max_{j \neq i} \psi_j(\theta_j)\}. \end{cases}$$

- (R) implies $\psi_i(\theta_i) > \psi_i(\tilde{\theta}_i)$ whenever $\theta_i > \tilde{\theta}_i$. Hence, for all $\theta_i > \tilde{\theta}_i$,

$$q_i(\theta_i, \theta_{-i}) \geq q_i(\tilde{\theta}_i, \theta_{-i}) \quad \text{and therefore} \quad \overline{q}_i(\theta_i) \geq \overline{q}_i(\tilde{\theta}_i).$$

Regularity and pointwise maximum

Regularity assumption: For all $i \in N$, the distribution F_i is such that

$$\psi_i(\theta_i) \text{ is strictly increasing in } \theta_i. \tag{R}$$

- Ignoring the monotonicity constraint on $\overline{q}_i(\theta_i)$, the objective function is maximized if q maximizes $\sum_{i \in N} q_i(\theta)\psi_i(\theta_i)$ for every possible θ, i.e.,

$$q_i(\theta) = \begin{cases} 1 & \text{if } \psi_i(\theta_i) > \max\left\{0, \max_{j \neq i} \psi_j(\theta_j)\right\}, \\ 0 & \text{if } \psi_i(\theta_i) < \max\left\{0, \max_{j \neq i} \psi_j(\theta_j)\right\}. \end{cases}$$

- (R) implies $\psi_i(\theta_i) > \psi_i(\tilde{\theta}_i)$ whenever $\theta_i > \tilde{\theta}_i$. Hence, for all $\theta_i > \tilde{\theta}_i$,

$$q_i(\theta_i, \theta_{-i}) \geq q_i(\tilde{\theta}_i, \theta_{-i}) \quad \text{and therefore} \quad \overline{q}_i(\theta_i) \geq \overline{q}_i(\tilde{\theta}_i).$$

- Under (R) the solution to the unconstrained problem satisfies the monotonicity constraint and thus also solves the constrained problem.

The main result

Proposition

Assume (R). A mechanism maximizes the seller's expected revenue if and only if it Bayesian implements (q^*, t) *such that for all* $i \in N$ *and* $\theta_i \in [\underline{\theta}_i, \overline{\theta}_i]$

$$q_i^*(\theta) = \begin{cases} 1 & \textit{if } \psi_i(\theta_i) > \max\{0, \max_{j \neq i} \psi_j(\theta_j)\}, \\ 0 & \textit{if } \psi_i(\theta_i) < \max\{0, \max_{j \neq i} \psi_j(\theta_j)\} \end{cases} \tag{5}$$

and

$$-\bar{t}_i(\theta_i) = \theta_i \bar{q}_i^*(\theta_i) - \int_{\underline{\theta}_i}^{\theta_i} \bar{q}_i^*(z)\, dz. \tag{6}$$

The resulting expected revenue is

$$E_\theta\Big[\max\{\psi_1(\theta_1), \psi_2(\theta_2), \ldots, \psi_n(\theta_n), 0\}\Big].$$

Implementation in dominant strategies

There are optimal transfers that induce even a dominant strategy equilibrium.

Define $y_i(\theta_{-i}) := \inf\left\{\theta_i \in [\underline{\theta}_i, \overline{\theta}_i] \;\middle|\; \psi_i(\theta_i) \geq 0 \text{ and } \forall j \neq i, \psi_i(\theta_i) \geq \psi_j(\theta_j)\right\}$.

- Given θ_{-i}, $y_i(\theta_{-i})$ is the lowest type θ_i such that $q_i^*(\theta) = 1$.

Implementation in dominant strategies

There are optimal transfers that induce even a dominant strategy equilibrium.

Define $y_i(\theta_{-i}) := \inf\left\{\theta_i \in [\underline{\theta}_i, \overline{\theta}_i] \;\middle|\; \psi_i(\theta_i) \geq 0 \text{ and } \forall j \neq i, \psi_i(\theta_i) \geq \psi_j(\theta_j)\right\}$.

- Given θ_{-i}, $y_i(\theta_{-i})$ is the lowest type θ_i such that $q_i^*(\theta) = 1$.

Proposition

Assume (R). The direct mechanism with allocation rule q^* and payment rule

$$-t_i(\theta) = \begin{cases} y_i(\theta_{-i}) & \text{if } q_i^*(\theta) = 1, \\ 0 & \text{if } q_i^*(\theta) = 0. \end{cases}$$

$\forall i \in N$ *is* ***dominant strategy*** *incentive compatible and revenue maximizing.*

Implementation in dominant strategies

There are optimal transfers that induce even a dominant strategy equilibrium.

Define $y_i(\theta_{-i}) := \inf\left\{\theta_i \in [\underline{\theta}_i, \overline{\theta}_i] \;\middle|\; \psi_i(\theta_i) \geq 0 \text{ and } \forall j \neq i, \psi_i(\theta_i) \geq \psi_j(\theta_j)\right\}$.

- Given θ_{-i}, $y_i(\theta_{-i})$ is the lowest type θ_i such that $q_i^*(\theta) = 1$.

Proposition

Assume (R). The direct mechanism with allocation rule q^* and payment rule

$$-t_i(\theta) = \begin{cases} y_i(\theta_{-i}) & \text{if } q_i^*(\theta) = 1, \\ 0 & \text{if } q_i^*(\theta) = 0. \end{cases}$$

$\forall i \in N$ *is* ***dominant strategy*** *incentive compatible and revenue maximizing.*

This mechanism is also **ex post** individually rational.

Proof.

The optimal allocation rule q^* given in (5) is equivalent to

$$q_i^*(\theta) = \begin{cases} 1 & \text{if } \theta_i > y_i(\theta_{-i}), \\ 0 & \text{if } \theta_i < y_i(\theta_{-i}). \end{cases}$$

Proof.

The optimal allocation rule q^* given in (5) is equivalent to

$$q_i^*(\theta) = \begin{cases} 1 & \text{if } \theta_i > y_i(\theta_{-i}), \\ 0 & \text{if } \theta_i < y_i(\theta_{-i}). \end{cases}$$

Define the payment rule as

$$-t_i(\theta) = \theta_i q_i^*(\theta) - \int_{\underline{\theta}_i}^{\theta_i} q_i^*(z, \theta_{-i}) dz.$$

This is an optimal payment rule since taking the expectation $E_{\theta_{-i}}[\cdot]$ gives (6).

Proof.

The optimal allocation rule q^* given in (5) is equivalent to

$$q_i^*(\theta) = \begin{cases} 1 & \text{if } \theta_i > y_i(\theta_{-i}), \\ 0 & \text{if } \theta_i < y_i(\theta_{-i}). \end{cases}$$

Define the payment rule as

$$-t_i(\theta) = \theta_i q_i^*(\theta) - \int_{\underline{\theta}_i}^{\theta_i} q_i^*(z, \theta_{-i}) dz.$$

This is an optimal payment rule since taking the expectation $E_{\theta_{-i}}[\cdot]$ gives (6).

$$-t_i(\theta) = \begin{cases} \theta_i - \int_{y_i(\theta_{-i})}^{\theta_i} 1 dz = y_i(\theta_{-i}) & \text{if } \theta_i > y_i(\theta_{-i}), \\ 0 & \text{if } \theta_i < y_i(\theta_{-i}). \end{cases}$$

Proof.

The optimal allocation rule q^* given in (5) is equivalent to

$$q_i^*(\theta) = \begin{cases} 1 & \text{if } \theta_i > y_i(\theta_{-i}), \\ 0 & \text{if } \theta_i < y_i(\theta_{-i}). \end{cases}$$

Define the payment rule as

$$-t_i(\theta) = \theta_i q_i^*(\theta) - \int_{\underline{\theta}_i}^{\theta_i} q_i^*(z, \theta_{-i}) dz.$$

This is an optimal payment rule since taking the expectation $E_{\theta_{-i}}[\cdot]$ gives (6).

$$-t_i(\theta) = \begin{cases} \theta_i - \int_{y_i(\theta_{-i})}^{\theta_i} 1 dz = y_i(\theta_{-i}) & \text{if } \theta_i > y_i(\theta_{-i}), \\ 0 & \text{if } \theta_i < y_i(\theta_{-i}). \end{cases}$$

The utility of buyer i with valuation θ_i who reports $\tilde{\theta}_i$ is

$$\theta_i q_i^*(\tilde{\theta}_i, \theta_{-i}) + t_i(\tilde{\theta}_i, \theta_{-i}) = \begin{cases} \theta_i - y_i(\theta_{-i}) & \text{if } \tilde{\theta}_i > y_i(\theta_{-i}), \\ 0 & \text{if } \tilde{\theta}_i < y_i(\theta_{-i}). \end{cases}$$

Proof.

The optimal allocation rule q^* given in (5) is equivalent to

$$q_i^*(\theta) = \begin{cases} 1 & \text{if } \theta_i > y_i(\theta_{-i}), \\ 0 & \text{if } \theta_i < y_i(\theta_{-i}). \end{cases}$$

Define the payment rule as

$$-t_i(\theta) = \theta_i q_i^*(\theta) - \int_{\underline{\theta}_i}^{\theta_i} q_i^*(z, \theta_{-i}) dz.$$

This is an optimal payment rule since taking the expectation $E_{\theta_{-i}}[\cdot]$ gives (6).

$$-t_i(\theta) = \begin{cases} \theta_i - \int_{y_i(\theta_{-i})}^{\theta_i} 1 dz = y_i(\theta_{-i}) & \text{if } \theta_i > y_i(\theta_{-i}), \\ 0 & \text{if } \theta_i < y_i(\theta_{-i}). \end{cases}$$

The utility of buyer i with valuation θ_i who reports $\tilde{\theta}_i$ is

$$\theta_i q_i^*(\tilde{\theta}_i, \theta_{-i}) + t_i(\tilde{\theta}_i, \theta_{-i}) = \begin{cases} \theta_i - y_i(\theta_{-i}) & \text{if } \tilde{\theta}_i > y_i(\theta_{-i}), \\ 0 & \text{if } \tilde{\theta}_i < y_i(\theta_{-i}). \end{cases}$$

$\Rightarrow$ Reporting $\tilde{\theta}_i = \theta_i$ maximizes i's utility for all θ_{-i}. □

The Symmetric Case

Suppose buyers are ex ante symmetric: $\underline{\theta}_i = \underline{\theta}$, $\overline{\theta}_i = \overline{\theta}$ and $F_i = F$ for all i.

The Symmetric Case

Suppose buyers are ex ante symmetric: $\underline{\theta}_i = \underline{\theta}$, $\overline{\theta}_i = \overline{\theta}$ and $F_i = F$ for all i.

- Hence, $\psi_i = \psi$ for all i.
- Assuming (R), q^* allocates to the highest θ_i, provided that $\psi(\theta_i) > 0$.
- Define

$$r^* := \begin{cases} \psi^{-1}(0) & \text{if } \psi(\underline{\theta}) < 0, \\ 0 & \text{if } \psi(\underline{\theta}) \geq 0. \end{cases}$$

  Hence, $y_i(\theta_{-i}) = \max\{r^*, \max_{j \neq i} \theta_j\}$.

The Symmetric Case

Suppose buyers are ex ante symmetric: $\underline{\theta}_i = \underline{\theta}$, $\overline{\theta}_i = \overline{\theta}$ and $F_i = F$ for all i.

- Hence, $\psi_i = \psi$ for all i.
- Assuming (R), q^* allocates to the highest θ_i, provided that $\psi(\theta_i) > 0$.
- Define

$$r^* := \begin{cases} \psi^{-1}(0) & \text{if } \psi(\underline{\theta}) < 0, \\ 0 & \text{if } \psi(\underline{\theta}) \geq 0. \end{cases}$$

Hence, $y_i(\theta_{-i}) = \max\{r^*, \max_{j \neq i} \theta_j\}$.

Proposition

Suppose (R) and bidders are ex ante symmetric. Then a ***second-price auction*** *with* ***reserve price*** r^* *is a revenue maximizing mechanism.*

The Symmetric Case

Suppose buyers are ex ante symmetric: $\underline{\theta}_i = \underline{\theta}$, $\overline{\theta}_i = \overline{\theta}$ and $F_i = F$ for all i.

- Hence, $\psi_i = \psi$ for all i.
- Assuming (R), q^* allocates to the highest θ_i, provided that $\psi(\theta_i) > 0$.
- Define

$$r^* := \begin{cases} \psi^{-1}(0) & \text{if } \psi(\underline{\theta}) < 0, \\ 0 & \text{if } \psi(\underline{\theta}) \geq 0. \end{cases}$$

  Hence, $y_i(\theta_{-i}) = \max\left\{r^*, \max_{j \neq i} \theta_j\right\}$.

Proposition

Suppose (R) and bidders are ex ante symmetric. Then a ***second-price auction*** *with* ***reserve price*** *r^* is a revenue maximizing mechanism.*

- SPA with reserve price r: if the highest bid is below r, seller keeps object; otherwise, winner pays maximum of r and the second highest bid.
- If $\underline{\theta} \geq \frac{1}{f(\underline{\theta})}$, $r^* = 0$. $\Rightarrow$ standard second-price auction, ex post efficiency