You have 90 minutes to solve the following two exercises!

1. **Mechanism Design with Money**

A government decides whether to built a new airport. The cost of building the airport is $c > 0$. There are $n \geq 2$ citizens affected by the decision. Citizen i's utility is given by

$$\theta_i g - t_i, \tag{1}$$

where $g \in \{0, 1\}$ denotes the governments decision (with $g = 1$ if and only if the aiport is built), θ_i is citizen i's valuation for the airport, and t_i is the transfer citizien i pays to the government.

Citizen i's valuation θ_i is drawn from the distribution F_i on the interval $[\underline{\theta}, \overline{\theta}]$, where F_i admits an everywhere strictly positive density f_i. The citizen's valuations are mutually independent. Denote $F(\theta) = \Pi_i F_i(\theta_i)$ the joint distribution of the type vector $\theta = (\theta_1, \ldots, \theta_n)$, and f its density function.

The citizen's transfers are used to finance the airport. There are no outside sources of money, thus the government faces the budget constraint

$$c \leq \sum_i t_i \qquad \text{if} \quad g = 1 \tag{2}$$

A direct mechanism (q, t) specifies for each type profile $\theta = (\theta_1, \ldots, \theta_n)$ a probability $q(\theta)$ of building the airport, and a vector of transfers $t(\theta) = (t_1(\theta), \ldots, t_n(\theta))$.

(a) Describe the set of socially optimal allocation(s) (q^*, t^*).

(b) Show that the ex-ante expected budget surplus (that is revenue minus costs) from any direct mechanism (q, t) satisfying DIC is given by

$$\int_\Theta q(\theta) \left[\sum_{i=1}^{n} \left(\theta_i - \frac{1 - F(\theta_i)}{f(\theta_i)} \right) - c \right] f(\theta) \mathrm{d}\theta - \sum_{i=1}^{n} U_i(\underline{\theta}) \tag{3}$$

where $U_i(\theta_i) = \mathbb{E}_{\theta_{-i}}(\theta_i q(\theta_i, \theta_{-i}) - t_i(\theta_i, \theta_{-i}))$.
Hint: use the envelope-formula for DIC-implementable mechanisms.

Define the 'pivot' mechanism (q^P, t^P) as follows: $q^P(\theta) = q^*(\theta)$ and

$$t_i^P(\theta) = \underline{\theta} q^*(\underline{\theta}, \theta_{-i}) + (q^*(\theta) - q^*(\underline{\theta}, \theta_{-i})) \left(c - \sum_{j \neq i} \theta_j \right) \tag{4}$$

(c) Prove that the pivot mechanism is DIC and IR.

(d) Prove that there is no DIC and IR mechanism that implements q^* and has larger ex-ante expected budget surplus than the pivot mechanism

(e) Briefly explain why the result in (d) extends to mechanisms that are Bayesian incentive compatible.

(f) Prove that for $c \in (n\underline{\theta}, n\overline{\theta})$ there is no incentive compatible and individually rational mechanism that implements a socially efficient allocation.

2. There are three alternatives, a, b and c, and three voters, $1, 2$ and 3, with strict preferences over the three alternatives.

   (a) Suppose ordinal rankings are given as follows:

$$a \succ_1 b \succ_1 c$$
$$b \succ_2 a \succ_2 c$$
$$c \succ_3 b \succ_3 a$$

   Which alternative is the Condorcet winner? Give an example of distinct ordinal preferences for each voter, such that alternative a is the Condorcet winner, and another example such that there is no Condorcet winner.

   (b) Now suppose preferences are single-peaked according to the linear order $a \geq b \geq c$. Which ordinal rankings can occur?

   From now on assume voters' preferences are single-peaked as described in (b). Further assume preferences are private information and each ordinal ranking (as determined in (b)) is equally likely. Cardinal preferences are such that a voter derives utility 1 from her most preferred alternative, utility 1/3 from the intermediate alternative, and utility zero from her least preferred alternative.

   (c) Consider the following sequential voting procedure. First, voters decide between alternatives a and c with simple majority. Next, voters decide (again with simple majority) whether to adopt the 'winner' of stage one or alternative b. Show that sincere voting is an equilibrium.

   (d) Consider the voting procedure from part (c), but now assume the first round is between alternatives a and b, and the winner goes against alternative c. Show that there is an equilibrium where everyone votes sincerely, with the exception of a voter type who strictly prefers alternative c: this type votes for a in the first round and sincerely in the second round. For which realization of individual preferences does this voting procedure fail to select the Condorcet winner?