Microeconomics II, Summer 2015 Second Exam, September 17, 2015

1. **Many-to-one matching**
   (a) Give the definitions of a matching and of a stable matching for a many-to-one matching market.
   (b) Consider a many-to-one matching market with 2 firms and 3 workers with the following preferences:

$$\begin{aligned}
&F_1 : \{w_1, w_2\}, \{w_2, w_3\}, \{w_1, w_3\}, \{w_3\}, \{w_2\}, \{w_1\} \\
&F_2 : \{w_3\}, \{w_2\} \\
&w_1 : F_1, F_2 \\
&w_2 : F_2 \\
&w_3 : F_1, F_2.
\end{aligned}$$

   Compute a stable matching.

   Consider now a general many-to-one matching market with strict preferences. The **firm-proposing DA** for this setting can be described as follows:

   In the first step, each firm proposes to its most preferred set of workers, and each worker rejects all but the most preferred acceptable firm that proposes to him. In each subsequent step, each firm proposes to its most preferred set of workers that includes all of those workers to whom it previously proposed to and who have not yet rejected it, but does not include any workers who have previously rejected it. Each worker rejects all but the most preferred acceptable firm that has proposed so far.

   The algorithm stops after any step in which there are no rejections, at which point each firm is matched to the set of workers to which it has issued proposals that have not been rejected.

   Let $Ch_F(S) = \{S' \subseteq S | \forall S'' \subseteq S,\ S' \succeq_F S''\}$. Preferences are *substitutable* if for all S and $w, w' \in S$, $w \in Ch_F(S)$ implies $w \in Ch_F(S \setminus \{w'\})$.

   (c) Suppose preferences are substitutable. Show that the firm-proposing DA yields a stable matching with respect to reported preferences.
   (d) Show by example that without substitutable preferences no stable matching might exist.
   (e) Consider the direct mechanism where each agent submits his preferences, the designer runs the firm-proposing DA on the reported preferences and implements the resulting matching. Suppose that all agents have substitutable preferences and can submit only substitutable preferences.

   Show by example that it is not a dominant strategy for firms to report truthfully.

2. **Mechanism Design**

There is a consumer who thinks about buying one unit of a good that has value 1 to him from one of n potential sellers. Each seller $i \in \{1, \ldots, n\}$ can produce the good at privately known cost θ_i. Each θ_i is independently drawn from a distribution F_i on $[0, 1]$ with density f_i. The virtual cost $\phi_i(\theta_i) := \theta_i + \frac{F_i(\theta_i)}{f_i(\theta_i)}$ is strictly increasing. If seller i produces the good with probability q_i and receives monetary transfer t_i, then his utility is $t_i - q_i\theta_i$.

Suppose the consumer wants to use an optimal mechanism that maximize his utility. A direct mechanism consists of an allocation rule $q(\theta) \in [0, 1]^n$ such that $\sum_{i=1}^n q_i(\theta) \leq 1$ and a payment rule $t(\theta) \in \mathbb{R}^n$. Given reports θ, seller i is paid $t_i(\theta)$ and produces with probability $q_i(\theta)$. Each seller i has outside option $\hat{u}_i(\theta_i) = 0$. The consumer's optimization problem can be stated as

$$\max_{(q,t)} \; E_\theta\left[\sum_{i=1}^n q_i(\theta) - \sum_{i=1}^n t_i(\theta)\right]$$

subject to Bayesian incentive compatibility and interim individual rationality for each i. Let $\overline{q}_i(\theta_i) := E_{\theta_{-i}}\big[q_i(\theta)\big]$ and $\overline{t}_i(\theta_i) := E_{\theta_{-i}}\big[t_i(\theta)\big]$ and define $U_i(\theta_i) = \overline{t}_i(\theta_i) - \overline{q}_i(\theta_i)\theta_i$.

(a) Use the Envelope Theorem to show formally that if (q, t) is Bayesian incentive compatible, then

$$\overline{t}_i(\theta_i) = \theta_i\overline{q}_i(\theta_i) + \int_{\theta_i}^1 \overline{q}_i(z)dz + U_i(1). \tag{IC1}$$

(b) Show that if (q, t) is Bayesian incentive compatible, the ex ante expected transfer to seller i can be written as $E_\theta\big[t_i(\theta)\big] = E_\theta\big[q_i(\theta)\phi_i(\theta_i)\big] + U_i(1)$.

Our characterization result from the lecture implies that (q, t) is Bayesian incentive compatible for i if and only if (IC1) holds and $\overline{q}_i(\theta_i)$ is non-increasing in θ_i.

(c) Solve the consumer's optimization problem.

Suppose now that the goods produced are of different qualities. Specifically, a seller of type θ_i produces a good of quality θ_i. The buyer values quality: an object of quality θ_i has value $\frac{3}{2}\theta_i$ to him.

(d) Describe the efficient allocation rule. Can it be implemented in a Bayesian incentive compatible mechanism?

(e) Which Bayesian incentive compatible mechanism maximizes aggregate expected utility? A sketch of the proof is sufficient.