The Optimal Allocation of Prizes

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Literature

1. Moldovanu & Sela: *AER* 2001, *JET* 2006
2. Moldovanu, Sela, Shi: *JPE* 2007, *Ec.Inquiry* 2011
3. Hoppe, Moldovanu, Sela: *RES* 2009
4. Mueller & Schotter, *JEEA* 2010
5. Kosfeld & Neckermann, *AEJ Micro* 2011
6. Boudreau et al., *RAND* 2016

The Model I

- Contest with n players where each player j makes an effort e_j.
- Efforts are submitted simultaneously. An effort e_j causes a cost of e_j/a_j, where a_j is an ability parameter.
- The ability (or *type*) of contestant j is private information to j.
- Abilities are drawn independently of each other from $[0, 1]$,according to a distribution F that is common knowledge.
- F has a continuous density $f = dF > 0$.

The Model II

- The designer can allocate n prizes: $0 \leq V_1 \leq V_2 \leq \cdots \leq V_n$.
- The contestant with the highest effort wins the first prize V_n, the contestant with the second highest effort wins the second prize V_{n-1}, and so on until all the prizes are allocated.
- The payoff of contestant i who has ability a_i and submits effort e_i is $V_j - e_i/a_i$ if i wins prize j.´
- Each contestant i chooses her effort in order to maximize her expected utility (given the other competitors' efforts and the values of the different prizes).
- The contest designer determines the number and size of prizes in order to maximize total expected effort $\sum_{i=1}^{n} e_i$.

Order Statistics I

1. $A_{k,n}$ denotes k-th order statistic out of n independent variables independently distributed according to F. $A_{n,n}$ is the highest order statistic, and so on...
2. $F_{k,n}$ denotes the distribution of $A_{k,n}$, and $f_{k,n}$ denotes its density;
3. $$F_{k,n}(s) = \sum_{i=k}^{n} \binom{n}{i} [F(s)]^i [1 - F(s)]^{n-i}$$
4. $E(k, n)$ denotes the expected value of $A_{k,n}$, where we set $E(0, n) = 0$.

Order Statistics II

- F_i^n denotes the probability that a player's type s ranks exactly i-th lowest among n random variables distributed according to F
- We have:

$$F_i^n(s) = \frac{(n-1)!}{(i-1)!(n-i)!}[F(s)]^{i-1}[1-F(s)]^{n-i},\ i = 1, 2, ..., n.$$

and

$$F_i^n(s) = F_{i-1,n-1}(s) - F_{i,n-1}(s)$$

where $F_{n,n-1}(s) \equiv 0$ and $F_{0,n-1}(s) \equiv 1$ for all $s \in [0,1]$

Equilibrium Derivation I

- We focus here on a symmetric equilibrium: let $\beta(a)$ denote the strategy for the player with type a.
- Assuming a symmetric equilibrium in **strictly increasing** strategies, and applying the revelation principle, we can formulate the player's optimization problem as follows:
- Player j with ability a chooses to behave as an agent with ability s in order to solve the following problem:

$$\max_s \sum_{i=1}^{n} F_i^n(s) V_i - \frac{\beta(s)}{a}$$

- Equivalently:

$$\max_s \sum_{i=1}^{n} \left[F_{i-1,n-1}(s) - F_{i,n-1}(s)\right] V_i - \frac{\beta(s)}{a}.$$

Equilibrium Derivation II

- In equilibrium, the above maximization problem must be solved by $s = a$.
- Boundary condition $\beta(0) = 0$
- The solution of the resulting differential equation is:

$$\beta(a) = \int_0^a x \left\{ \begin{array}{c} -f_{1,n-1}(x)V_1 + \\ \sum_{i=2}^{n-1} \left[f_{i-1,n-1}(x) - f_{i,n-1}(x) \right] V_i + f_{n-1,n-1}(x)V_n \end{array} \right\} dx.$$

Total Effort I

- The expected total effort is given by

$$E_{total} = n \int_0^1 \beta(a) f(a) da.$$

- Note that

$$\begin{aligned} & n \int_0^1 \left[\int_0^a x f_{r,n-1}(x) dx \right] f(a) da \\ = \; & n \left[F(a) \int_0^a x f_{r,n-1}(x) dx \right]_0^1 - n \int_0^1 F(a) a f_{r,n-1}(a) da \\ = \; & n \int_0^1 a \left(1 - F(a)\right) f_{r,n-1}(a) da, \end{aligned}$$

where the first equality follows from integration by parts.

Total Effort II

- We further observe that

$$n\left(1-F\left(a\right)\right)f_{r,n-1}(a)=\left(n-r\right)f_{r,n}(a)$$

.

- Therefore, we have

$$n\int_0^1\left[\int_0^a xf_{r,n-1}(x)dx\right]f(a)da=\left(n-r\right)E\left(r,n\right)$$

- The expected total effort becomes

$$E_{total}(V_1,V_2,...,V_n)=\sum_{i=1}^{n}\left[\left(n-i+1\right)E\left(i-1,n\right)-\left(n-i\right)E\left(i,n\right)\right]V_i.$$

The Optimal Prize Structure

- The designer has a budget $P < \infty$. His problem is:

$$\max_{\{V_1,\ldots,V_n\}} E_{total}(V_1, V_2, \ldots, V_n)$$
$$s.t. \quad : \quad \sum_{i=1}^{n} V_i \leq P.$$

Theorem

The optimal prize structure is $V_n = P$ and $V_i = 0$ for all $i < n$.

The Optimal Prize Structure: Proof

Marginal effect of V_i :

$$\begin{aligned}\frac{\partial E_{total}}{\partial V_i} &= (n-i+1)\,E\,(i-1,n) - (n-i)\,E\,(i,n) \\ &= E(i,n) - \ (n-i+1)\,[E\,(i,n) - E\,(i-1,n)],\ 1 \leq i \leq n\end{aligned}$$

Observe that :

$$\frac{\partial E_{total}}{\partial V_n} = E\,(n-1,n) \geq 0$$

and that

$$\frac{\partial E_{total}}{\partial V_n} - \frac{\partial E_{total}}{\partial V_i} = E\,(n-1,n) - E(i,n) + \ (n-i+1)\,[E\,(i,n) - E\,(i-1,n)] \geq 0$$

for $1 \leq i < n$.

Thus, the designer optimally rewards only the player with the highest effort.

Single Crossing Properties I (Moldovanu&Sela, *JET* 2006)

Theorem

Consider a contest with n contestants and with equal prizes. For any number of prizes p, r such that $n \geq r > p$, the equilibrium effort functions $\beta_{n,r}(c)$ and $\beta_{n,p}(c)$ are single-crossing. That is, there exists a unique $c^* = c^*(n, r, p) \in (0, 1)$ such that

$$\begin{aligned} \beta_{n,r}(c^*) &= \beta_{n,p}(c^*) \\ \beta_{n,p}(c) &> \beta_{n,r}(c) \text{ for all } c \in [0, c^*) \\ \beta_{n,p}(c) &< \beta_{n,r}(c) \text{ for all } c \in (c^*, 1) \end{aligned}$$

Single Crossing Properties II

Theorem

Consider a contest with p equal prizes. For any numbers of contestants n, k such that $n > k \geq p$, the equilibrium effort functions $\beta_{n,p}(c)$ and $\beta_{k,p}(c)$ are single-crossing. That is, there exists a unique $c^* = c^*(n,k,p) \in (0,1)$ such that

$$\begin{aligned} \beta_{n,p}(c^*) &= \beta_{k,p}(c^*) \\ \beta_{n,p}(c) &> \beta_{k,p}(c) \text{ for all } c \in [0, c^*) \\ \beta_{n,p}(c) &< \beta_{k,p}(c) \text{ for all } c \in (c^*, 1) \end{aligned}$$

Majorization

Definition

A vector $\mathbf{x} \in \mathbb{R}^n$ is said to majorize a vector $\mathbf{y} \in \mathbb{R}^n$ $(\mathbf{x} \succsim \mathbf{y})$ if

$$\sum_{i=1}^{n} x_i = \sum_{i=1}^{n} y_i$$

and if their increasing rearrangements $\overrightarrow{\mathbf{x}}$ and $\overrightarrow{\mathbf{y}}$ satisfy

$$\sum_{i=k}^{n} \overrightarrow{x}_i \geq \sum_{i=k}^{n} \overrightarrow{y}_i \text{ for all } 1 \leq k \leq n$$

A function $\Psi : \mathbb{R}^n \to \mathbb{R}$ such that $\Psi(\mathbf{x}) \geq \Psi(\mathbf{y})$ if $\mathbf{x} \succsim \mathbf{y}$ is called *Schur-Convex.*

Example

$(0, 0, 1) \succsim (1/3, 1/3, 1/3)$.

Generalization to Different Prizes

Theorem

For any vector of prizes $\mathbf{V}$ *and* $\mathbf{W}$ *such that the corresponding effort bidding functions* $\beta_{\mathbf{V}}$ *and* $\beta_{\mathbf{W}}$ *are single-crossing (with high types bidding more in* $\beta_{\mathbf{V}}$ *and lower types bidding more in* $\beta_{\mathbf{W}}$)*.*

Theorem

Total effort is a Schur-Convex function of prizes, e.g., total effort under prizes $\mathbf{V}$ *is higher than total effort under prizes* $\mathbf{W}$ *if* $\mathbf{V} \succsim \mathbf{W}$*.In particular, the most dispersed possible vector of prizes (a unique prize !) is optimal.*

Experimental Evidence (Mueller&Schotter, *JEEA*, 2010)

- 2×2 experimental design
- Contest with linear costs. Treatments: LC-1 (Prize), LC-2. LC-1 is optimal.
- Contest with quadratic costs. Treatments: QC-1, QC-2. QC-2 is optimal.
- Results about effort agree well with theory at the aggregate level.
- Disaggregate data (individual effort): over-shooting at the top, drop-outs at the bottom.
- Attribute the above to loss aversion.

Field Evidence I (Boudreau et al., *RAND* 2016)

- 755 algorithm contests at TopCoder, Inc. with 2775 participants
- Each contest ("room") has min. 15, max. 20 participants.
- Skill distribution estimated by TopCoder's *SkillRating*
- Tests response of contestants to increasing the number of participants in each "room"
- Results agree well with theory: zero response at the bottom, negative falling response for low skill, increasing positive response for high skill.

Field Evidence II (Kosfeld&Neckerrmann, *AEJ* 2011)

- Field experiment for NGO: search and entry of data in database
- 16 contests, 9.4 contestant per group on average.
- Treatment 1: fixed pay per hour (effort independent)
- Treatment 2: fixed pay per hour + symbolic awards for two top achievers in each group.
- Hypothesis: symbolic awards represent status prizes (Moldovanu, Sela, Shi, *JPE* 2007)
- Results: average effort about 12% higher with symbolic awards.

Matching Contests I (Hoppe, Moldovanu, Sela *RES* 2009)

- We consider measures 1 of "men" and "women".
- Each man has attribute x, each woman attribute y. Attributes are private info.
- If a man and a woman are matched, the utility for each is the product of their attributes.
- Total output from a match between agents with types x and y is $2xy$.
- Attributes X, Y are independently distributed over $[0, 1]$ according to distributions F (men) and G (women), respectively.
- $F(0) = G(0) = 0$, that F and G have continuous densities, $f > 0$ and $g > 0$, and finite first and second moments.

Matching Contests II

- Each agent sends a costly signal b, and signals are submitted simultaneously.
- Agents on each side are ranked according to signals, and get matched assortatively: man with the highest signal is matched with the woman with the highest signal, etc....
- Agents with same signals are randomly matched to the corresponding partners.
- The net utility of a man x that is matched to a woman y after sending signal b is given by $xy - b$ (and similarly for women).

Random versus Assortative Matching

- Random matching yields an expected total output (and welfare) of

$$2E(X)E(Y)$$

. No need to signal !

- Under assortative matching, a man x is matched with a woman $y = \psi(x)$, where

$$\psi(x) = G^{-1}F(x)$$

- Expected total output under assortative matching is given by

$$2\int_0^1 x\psi(x)f(x)dx$$

.

- To achieve assortative matching signals are needed !. This creates a better matching (more output) but also wastes utility.
- Is assortative matching based on signaling better than random matching in terms of total welfare ?

Assortative Matching Equilibrium

- Consider men's types $x, \hat{x}$, $x > \hat{x}$, with bids $\beta(x) > \beta(\hat{x})$.
- Type x is matched with $\psi(x)$, and $\hat{x}$ is matched with $\psi(\hat{x})$. Type x should not pretend that he is $\hat{x}$, and vice-versa. This yields:

$$\begin{aligned} x\psi(x) - \beta(x) &\geq x\psi(\hat{x}) - \beta(\hat{x}) \\ \hat{x}\psi(\hat{x}) - \beta(\hat{x}) &\geq \hat{x}\psi(x) - \beta(x) \end{aligned}$$

Combining and dividing by $x - \hat{x}$, gives:

$$\frac{\hat{x}\psi(x) - \hat{x}\psi(\hat{x})}{x - \hat{x}} \leq \frac{\beta(x) - \beta(\hat{x})}{x - \hat{x}} \leq \frac{x\psi(x) - x\psi(\hat{x})}{x - \hat{x}}$$

- Taking the limit $\hat{x} \to x$ gives $\beta'(x) = x\psi'(x)$. Together with $\beta(0) = 0$, this yields

$$\beta(x) = \int_0^x z\psi'(z)\, dz = x\psi(x) - \int_0^x \psi(t)dt$$

How Much Is Wasted In Equilibrium ?

- Note that:

$$\forall x, \int_0^x \psi(t)dt + \int_0^{\psi(x)} \varphi(t)dt = x\psi(x)$$

where $\varphi = \psi^{-1}$

- This yields:

$$\beta(x) + \gamma(\psi(x)) = 2x\psi(x) - x\psi(x) = x\psi(x)$$

- Half of output in each pair is wasted on signaling !

Random vs. Assortative Matching?

- Assortative matching with signaling is welfare superior (inferior) to random matching if:

$$\begin{aligned} E(X \cdot \psi(X)) &\geq [\leq]\ 2E(X) \cdot E(Y) \Leftrightarrow \\ E(X \cdot \psi(X)) &\geq [\leq]\ 2E(X) \cdot E(\psi(X)) \Leftrightarrow \\ \frac{Cov(X,\ \psi(X))}{EX \cdot E(\psi(X))} &\geq [\leq]\ 1 \end{aligned}$$

- For $F = G$ we obtain that assortative matching with signaling is welfare superior (inferior) to random matching if

$$\frac{Var(X)}{(EX)^2} \geq [\leq]\ 1 \Leftrightarrow \frac{STD(X)}{EX} \geq [\leq]\ 1$$