Microeconomics II (PhD)
Tutorial 5, May 30th, 2018

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Hand in written solutions *before* the tutorial on may 30th.
You may work in groups of at most two students.

Exercises:

1. Solve Exercise 23.E.3 in MWG.

2. *(Variant of an old exam question)*
   Consider an auction setting: a seller has a single object for sale, which he does not value. There are 2 bidders, who are privately informed about their valuations θ_i. Valuations are drawn independently, where $\theta_1 \sim U[0,1]$ and $\theta_2 \sim U[0,2]$. Each buyer has a utility function $u_i = p_i \cdot \theta_i + t_i$, where p_i denotes the probability that buyer i gets the object and t_i denotes the transfer he receives.

   Any auction has to be Bayesian incentive compatible and give each bidder type an interim expected utility of at least 0.

   You may use all results from the lecture without proof.

   (a) Compute the allocation rule of the revenue maximizing auction. Illustrate graphically why this allocation rule is inefficient.

   (b) Compute the interim expected utility of bidder 1 as a function of his type.

   (c) Suppose the seller has a commonly known reservation value of $\frac{1}{2}$. His utility is therefore given by $u_S = (1 - p_1 - p_2) \cdot \frac{1}{2} - t_1 - t_2$. Compute the allocation rule of the auction that is optimal for the seller.

   (d) Suppose that types are not drawn independently. Instead, types are distributed on $[0,1]^2$ according to the cdf $F(\theta_1, \theta_2) = \min\{\theta_1, \theta_2\}$ for $0 \leq \theta_1, \theta_2 \leq 1$ (i.e., types are perfectly correlated).

   How does an optimal auction look like? Check that incentive and participation constraints are fulfilled. Compute the interim expected utilities of the buyers.

   (e) Suppose now that each bidder has either a valuation of 1 or 2.

$$Prob(\theta_1 = 1, \theta_2 = 1) = Prob(\theta_1 = 2, \theta_2 = 2) = \frac{1}{4} + \varepsilon$$
$$Prob(\theta_1 = 1, \theta_2 = 2) = Prob(\theta_1 = 2, \theta_2 = 1) = \frac{1}{4} - \varepsilon$$

   for some $\varepsilon > 0$. Compute the optimal auction and the corresponding revenue.

3. There is one seller with two objects, and one buyer. The seller does not value the objects; the buyer values object k by θ^k ($k = 1, 2$) and getting both objects by $\theta^1 + \theta^2$.

   (a) Suppose that valuations are independently distributed and, for $k = 1, 2$,

   $$\theta^k = \begin{cases} 10 & \text{with probability } \frac{1}{2} \\ 22 & \text{with probability } \frac{1}{2}. \end{cases}$$

   What are the optimal prices and the corresponding revenue if the seller sells the objects separately? What is the optimal price and the corresponding revenue if the seller only sells the bundle?

   (b) Suppose that valuations are independently distributed and, for $k = 1, 2$,

   $$\theta^k = \begin{cases} 10 & \text{with probability } \frac{1}{2} \\ 50 & \text{with probability } \frac{1}{2}. \end{cases}$$

   What are the optimal prices and the corresponding revenue if the seller sells the objects separately? What is the optimal price and the corresponding revenue if the seller only sells the bundle?

   (c) Suppose the seller sets a price for each object and a price for the bundle of both objects. Determine the optimal prices if valuations are identically, independently, and uniformly distributed on $[0, 1]$.

   (d) Suppose valuations are independently distributed and, for $k = 1, 2$,

   $$\theta^k = \begin{cases} 1 & \text{with probability } \frac{1}{6} \\ 2 & \text{with probability } \frac{1}{2} \\ 4 & \text{with probability } \frac{1}{3} \end{cases}$$

   The expected revenue in the optimal deterministic mechanism is $\frac{29}{9}$.

   Suppose the seller offers the following menu: A lottery which yields with probability $\frac{1}{2}$ object 1 and nothing otherwise, a lottery which yields with probability $\frac{1}{2}$ object 2 and nothing otherwise, and getting the bundle of both objects for sure. Show that the seller can obtain a larger expected revenue offering this menu compared to the optimal deterministic mechanism.

4. You can use the following envelope theorem in this exercise.

Theorem (Milgrom and Segal, 2002)**.**
Let X be an arbitrary set, $T = [\underline{t}, \overline{t}]$,[1] *and $f : X \times T \to \mathbb{R}$. Denote*

$$V(t) = \sup_{x \in X} f(x,t) \tag{1}$$
$$X^*(t) = \{x \in X | f(x,t) = V(t)\}. \tag{2}$$

Suppose that $f(x,\cdot)$ is differentiable for all $x \in X$, $f_t(x,\cdot)$ is uniformly bounded and that $X^*(t) \neq \emptyset$ for almost all t. Then for any selection $x^*(t) \in X^*(t)$,

$$V(t) = V(\underline{t}) + \int_{\underline{t}}^{t} f_t(x^*(s), s) ds. \tag{3}$$

Consider the general mechanism design setting from the lecture, where $v_i(k, \theta_i)$ denotes the value of allocation k to agent i with type θ_i. Suppose that $\Theta_i = [\underline{\theta}_i, \overline{\theta}_i] \subset \mathbb{R}$ and that v_i is differentiable in θ_i for all k and the derivative is uniformly bounded. Given a direct revelation mechanism (k, t), let $U_i(\theta) = v_i(k(\theta), \theta_i) + t_i(\theta)$ be the utility of agent i if θ is the profile of types and all agents report truthfully.

(a) Show that if the direct revelation mechanism (k, t) is implementable in dominant strategies, then

$$U_i(\theta) = U_i(\underline{\theta}_i, \theta_{-i}) + \int_{\underline{\theta}_i}^{\theta_i} \frac{\partial v_i(k(s, \theta_{-i}), s)}{\partial \theta_i} ds. \tag{ICFOC}$$

Suppose that $v_i(k, \theta_i)$ has the single-crossing property: $\frac{\partial^2 v_i(k,\theta_i)}{\partial k \ \partial \theta_i}$ exists and is stricly positive for all $k \in K$ and $\theta_i \in \Theta_i$.

(b) Show that if the direct revelation mechanism (k, t) is implementable in dominant strategies, then $k(\theta_i, \theta_{-i})$ is weakly increasing in θ_i for all θ_{-i}.

(c) Show that any monotone mechanism that satisfies (ICFOC) is implementable in dominant strategies.

(d) Briefly discuss the relation of these results to the result that you saw in the lecture. (2-3 sentences)

(e) Show: If a direct revelation mechanism implements the value-maximizing allocation rule in dominant strategies, then it is a VCG mechanism.

[1] This result holds more generally, for example if $T \subset \mathbb{R}^n$ is convex.