Multi-object auctions (and matching with money)
Introduction

Many auctions have to assign multiple heterogeneous “objects” among a group of heterogeneous buyers.

Examples: Electricity auctions (HS C 18:00), auctions of government bonds, internet add auctions,...

Questions for the study of auction mechanisms:

- Efficiency - Are objects assigned to the “right” bidders?
- Stability - Does the mechanism yield “sufficiently high” revenues for the seller(s)?
- Incentives - Do bidders have an incentive to bid “straightforwardly”?

We’ll start by reconsidering single-object auctions in light of what we’ve learned in part III so far and then build up towards a model with many sellers and buyers.

Literature: Chapters 7 and 8 in RS (1990)
Suppose there is one seller \( n + 1 \) who has one indivisible object to sell and \( n \) bidders \( 1, \ldots, n \).

Agent \( i \in \{1, \ldots, n + 1\} \) has value \( \alpha_i \geq 0 \) for the object.

All agents have \textit{quasi-linear utility}.

⇒ An outcome can be described by a (monetary) payoff vector \( u = (u_1, \ldots, u_{n+1}) \in \mathbb{R}^{n+1} \).

Note: We allow for \textit{side-payments}, so that e.g. the winning bidder could make payments to losing bidders.

\begin{equation}
\nu(S) = \begin{cases}
0, & \text{if } n + 1 \notin S \\
\max_{i \in S \cap \{1, \ldots, n+1\}} \alpha_i, & \text{otherwise}
\end{cases}
\end{equation}

Worth of coalition \( S \subseteq N = \{1, \ldots, n + 1\} \), \( \nu(S) \), is maximum payoff achievable by \( S \).
Single-object auctions: A reconsideration

- An payoff vector $u$ is feasible, if $\sum_i u_i \leq v(N)$.
- A feasible payoff vector $u$ is pairwise stable, if
  (a) it is individually rational, i.e. $u_i \geq 0$ for all $i \in \{1, \ldots, n\}$ and $u_{n+1} \geq \alpha_{n+1}$, and
  (b) if there are no $i \in \{1, \ldots, n\}$, $p \in \mathbb{R}$, such that $p > u_{n+1}$ and $\alpha_i - p > u_i$.
- A feasible payoff vector $u$ is in the core, if $\sum_{i \in S} u_i \geq v(S)$, for all $S \subseteq N$.
- Remark: $u$ is pairwise stable if and only if it is in the core.
Single-object auctions: A reconsideration

Theorem

Suppose $\alpha_1 \geq \alpha_2 \geq \ldots \geq \alpha_n$ and let $\alpha^* = \max\{\alpha_2, \alpha_{n+1}\}$.

(i) If $\alpha_1 > \alpha^*$, the core is given by $\{u \in \mathbb{R}^{n+1} : u_{n+1} \in [\alpha^*, \alpha_1], u_1 = \alpha_1 - u_{n+1}, \text{ and } u_i = 0, \forall i \in \{2, \ldots, n\}\}$.

(ii) In any other case, the core is a singleton consisting of $(0, \ldots, 0, \nu(N))$.

Remarks:

- No side payments at stable allocations: winning bidder pays a price $u_{n+1}$ to the seller
- Set of stable prices = set of market-clearing prices
- Note close analogy to results for matching models without transfers
What about agents’ incentives to reveal their true $\alpha_i$s?

- There is no stable matching/auction mechanism which is strategyproof for all agents.
- The bidder optimal stable matching mechanism (i.e. second-price auction with reserve price $v_{n+1}$/Vickrey mechanism) is strategyproof for bidders.
- The seller optimal stable matching mechanism (i.e. first-price auction with secret reserve) is strategyproof for the seller.
Our main interest: Develop “good” auction mechanisms for the case of $m$ bidders and $n$ sellers

Before that we need to set our goals...

Two important requirements:

- Allocation of objects/sellers to bidders should be *efficient*.
- Buyers and sellers should not have an incentive to circumvent the mechanism, i.e. allocations should be *stable*.

In a first step, we’ll now study (a special case of) the *assignment game* and show that these goals can be met.
Assignment game

- Finite set of agents $N = P \cup Q$ where
  - $P$ is a set of potential buyers with $|P| = m$,
  - $Q$ is a set of sellers with $|Q| = n$

- Buyer $i \in P$ characterized by vector of valuations $(\alpha_{ij})_{j \in Q}$, where $\alpha_{ij} \geq 0$ is his value for object of seller $j \in Q$.

  **Assumption:** There is one seller $O \in Q$ such that $\alpha_{iO} = 0$ for all $i \in P$.

- Seller $j \in Q \setminus \{O\}$ owns one indivisible object; $O$ owns $m$ objects; all sellers attach a value of 0 to (all of their) objects

- Buyers have *unit demands:* If $i \in P$ acquires set of objects $A \subseteq Q$, utility from these objects is $\max_{j \in A} \alpha_{ij}$

- All agents have *quasi-linear utility*
Optimal assignments and feasible payoffs

An assignment is a matrix $x = x_{ij} \in [0, 1]^{n \times m}$. An assignment $x$ is deterministic, if $x_{ij} \in \{0, 1\}$ for all $i, j$.

The worth of coalition $S \subseteq N$, $\nu(S)$, is the maximum value $S$ can achieve by pairing $P$ and $Q$ agents.

Formally, $\nu(S)$ is the value function of the following optimization problem $\mathcal{P}_S$:

$$\max \sum_{i \in S \cap P, j \in S \cap Q} \alpha_{ij} x_{ij} \quad \text{s.t.} \quad \begin{align*}
(a) \quad & \sum_{j \in S \cap Q} x_{ij} \leq 1, \forall i \in P \\
(b) \quad & \sum_{i \in S \cap P} x_{ij} \leq 1, \forall j \in Q \setminus \{O\} \\
(c) \quad & x_{ij} \geq 0, \forall i, j \in S
\end{align*}$$
Optimal assignments and feasible payoffs

- An assignment \( x \) is **feasible** if it satisfies constraints (a) - (c) of \( \mathcal{P}_N \).
- A feasible assignment \( x \) is **optimal**, if \( \sum_{i,j} \alpha_{ij} x_{ij} = v(N) \).
  **Remark:** There exists an optimal assignment that is deterministic.
- A payoff vector \( (u, v) \in \mathbb{R}^m \times \mathbb{R}^n \) is **feasible**, if it is compatible with some feasible assignment \( x \), i.e.

\[
\sum_{i \in P} u_i + \sum_{j \in Q} v_j = \sum_{i \in P, j \in Q} \alpha_{ij} x_{ij}
\]

- A **feasible outcome** is a pair \( ((u, v); x) \) such that \( (u, v) \in \mathbb{R}^n \times \mathbb{R}^m \) is compatible with \( x \).
Stability

Definition

- A feasible payoff vector \((u, v)\) is *pairwise stable* if
  1. \(u_i \geq 0\) and \(v_j \geq 0\)
  2. \(u_i + v_j \geq \alpha_{ij}\) for all \((i, j) \in P \times Q\).

- An feasible payoff vector \((u, v) \in \mathbb{R}^m \times \mathbb{R}^n\) is in the *core* if
  \[\sum_{i \in S \cap P} u_i + \sum_{j \in S \cap Q} v_j \geq v(S)\]
  for all \(S \subseteq N\).

- A feasible outcome \(((u, v); x)\) is *pairwise stable (in the core)* if \((u, v)\) is compatible with \(x\) and \((u, v)\) is pairwise stable (in the core).

Theorem

*A feasible payoff vector is pairwise stable if and only if it is in the core.*
Stability: Existence

**Theorem**

A _stable outcome always exists in the assignment game._

**Remark:** Two possible proof strategies...

(i) Prove existence directly by appealing to _duality theory_ (that’s what we’ll do).

(ii) Introduce discrete money into the marriage model and then let size of smallest monetary unit converge to zero (see Kelso and Crawford (Econometrica, 1982)).
Consider the following optimization problem $\mathcal{P}^*$:
\[
\min \sum_i u_i + \sum_j v_j \quad s.t. \quad (a^*)u_i \geq 0, \forall i \in P, v_j \geq 0, \forall j \in Q
\]
\[
(b^*)u_i + v_j \geq \alpha_{ij}, \forall i \in P, j \in Q
\]

This is the dual of the problem $\mathcal{P}_N$ of finding an optimal assignment.

By a fundamental theorem of linear programming (see e.g. Vohra Ch. 4),

(i) $\mathcal{P}^*$ is solvable since $\mathcal{P}_N$ is solvable, and

(ii) if $(u, v)$ is a solution of $\mathcal{P}^*$ and $x$ is a solution of $\mathcal{P}_N$ then
\[
\sum_i u_i + \sum_j v_j = \sum_{i,j} \alpha_{ij}x_{ij}.
\]
\[\Rightarrow \quad \text{A stable outcome always exists!}\]
Stability: Structure

Theorem

(i) If $x$ is any optimal assignment, then it is compatible with any stable payoff vector $(u, v)$.

(ii) If $((u, v); x)$ is a stable outcome, $x$ is an optimal assignment.

(iii) If $((u, v); x)$ is a stable outcome and $x_{ij} = 1$ for some $i \in P$, $j \in Q$, then $u_i + v_j = \alpha_{ij}$.

(iv) If $((u, v); x)$ and $((u', v'); x')$ are two stable outcomes then $x'_{ij} = 1$ and $u'_i > u_i$ imply $v_j > v'_j$. 
Remark:

- By the first property, we can concentrate on payoffs when talking about feasible assignments.
- By the second property, stable outcomes are always efficient (this already follows from the core property).
- By the third property, there are no *side-payments* at stable outcomes.
- The fourth property is familiar *opposition of interests* result.
Stability: Structure

- Define the partial order $\geq_P$ on $\mathbb{R}^m \times \mathbb{R}^n$ by $(u', v') \geq_P (u, v)$ if and only if $u'_i \geq u_i$ for all $i \in P$, and $(u', v') >_P (u, v)$ if and only if $(u', v') \geq_P (u, v)$ and $(u', v') \neq (u, v)$.

- Given two payoff vectors $(u, v)$ and $(u', v')$, the smallest upper bound (greatest lower bound) of $(u, v)$ and $(u', v')$ with respect to $\geq_P$ is called the join (meet) of $(u, v)$ and $(u', v')$ and is denoted by $(u, v) \bigvee_P (u', v') ((u, v) \bigwedge_P (u', v'))$.

- A set of payoff vectors (i.e. a subset $\mathcal{P} \subseteq \mathbb{R}^m \times \mathbb{R}^n$) is a lattice w.r.t. $\geq_P$ if for any $(u, v), (u', v') \in \mathcal{P}$, $(u, v) \bigvee_P (u', v') \in \mathcal{P}$ and $(u, v) \bigwedge_P (u', v') \in \mathcal{P}$.

- A lattice $\mathcal{P} \subseteq \mathbb{R}^m \times \mathbb{R}^n$ is complete, if for any subset $Q \subseteq \mathcal{P}$, $\bigvee_P \{Q\} \in \mathcal{P}$ and $\bigwedge_P \{Q\} \in \mathcal{P}$.
Stability: Structure

**Theorem**

The set of stable payoff vectors is a complete lattice w.r.t. $\geq_P$. In particular, there exists a $P$-optimal stable payoff vector.

**Remark:**

- Can define $\geq_Q$ analogously and obtain that set of stable payoff vectors is a complete lattice w.r.t. $\geq_Q$.
- Easy to show that $(u, v) \lor_P (u', v') = (u, v) \land_Q (u', v')$ and $(u, v) \land_P (u', v') = (u, v) \lor_Q (u', v')$. 
  $\Rightarrow$ $P$-optimal stable payoff vector $= Q$-pessimal stable payoff vector.
So far, we have studied the assignment problem from a *cooperative view* in which buyers and sellers negotiate prices and ultimately (in a frictionless market) arrive at a stable outcome.

From basic microeconomics you are familiar with another view of this problem: There is a price for each object, agents are price takers, and in (competitive) equilibrium prices balance supply and demand.

⇒ What is the relationship between these two views of the problem?
Stability and Competitive Equilibria

Given a price vector \( p \in \mathbb{R}^n_+ \), define \( i \)'s demand by

\[
D_i(p) := \{ j \in Q : \alpha_{ij} - p_j = \max_k \alpha_{ik} - p_k \}.
\]

**Definition**

(i) A price vector \( p \) is *quasi-competitive* if there is a deterministic assignment \( x \) such that for all \( i \in P \), \( x_{ij} = 1 \) implies \( j \in D_i(p) \) (in this case we say that \( x \) is *compatible with* \( p \)).

(ii) A pair \((x, p)\) is a *competitive equilibrium* if \( x \) is compatible with \( p \) and for all \( j \in Q \) such that \( \sum_i x_{ij} = 0 \), \( p_j = 0 \).

(iii) A price vector \( p \) is *competitive* if there is some \( x \) such that \((x, p)\) is a competitive equilibrium.
**Theorem**

Let $x$ be some deterministic optimal assignment. An outcome $((u, v); x)$ is stable if and only if $(x, v)$ is a competitive equilibrium.

**Corollary**

There exists a minimum competitive price vector, i.e. an equilibrium price vector $p$ such that for any other equilibrium price vector $q$, $p_j \leq q_j$ for all $j \in Q$.

**Remark:** This equivalence does not carry over to more general settings (e.g. buyers and sellers with multiunit demand) (see Sotomayor (JET, 2007)).
Multi-object auction

With heterogenous objects an auction has to (simultaneously) determine

- *who gets which objects*, and
- *how much winning bidders pay*.

We now describe a generalization of the second-price auction (due to Demange et al. (JPE, 1986)) which yields a buyer optimal stable/competitive allocation.

**Assumption:** There is some $\Delta > 0$ such that all $\alpha_{ij}$s are multiples of $\Delta$. 
Multi-object auctions

Multiobject auction

Given some subset $B \subseteq P$ and a price vector $p$, let $D_B(p) := \bigcup_{i \in B} D_i(p)$ be the set of objects demanded by bidders in $B$.

**Theorem**

A price vector $p$ is quasi-competitive if and only if for all $B \subseteq P$ such that $D_B(p) \subseteq Q \setminus \{O\}$, $|D_B(p)| \geq |B|$.

**Remark:**

- This implies that if $p$ is not quasi-competitive, there exists an *overdemanded set of objects* $A \subseteq Q \setminus \{O\}$, that is, there is some $B \subseteq P$ such that $D_B(p) \subseteq A$ and $|B| > |A|$.
- This is a version of what is known as *Hall’s theorem*.
Outline of the auction: In each step, bidders report their demands given some (fixed) price vector and the auctioneer then raises prices for overdemanded sets of objects.

Algorithm initialized by setting $p_j(0) = 0$ for all $j \in Q$ and the $t + 1$st step ($t \geq 0$) proceeds as follows:

**Step** $t + 1$: If $p(t)$ is a quasi-competitive price, stop. Else, let $A$ be a minimal (in the subset order) overdemanded set and raise the price of each object $j \in A$ to $p_j(t) + \Delta$. Let the resulting price vector be denoted by $p(t + 1)$ and proceed to step $t + 2$. 
Theorem

Let $p$ be the price vector obtained from the multiobject auction mechanism. Then $p$ is the minimum quasi-competitive price and there is an assignment $x^*$ such that $(p, x^*)$ is a competitive equilibrium.

Note: By previous results, the auction can be used to identify the buyer-optimal stable outcome as well.
Multiobject auction: Incentives

- Auction is a *generalization of second-price auction*: Each winning bidder pays highest losing bid for his object.
- This may lead one to think that buyers have a dominant strategy of submitting valuations (the vector \( (\alpha_{ij})_{j \in Q} \)) truthfully...
- While this will turn out to be true, reasoning is more complicated since we have to check that no buyer can benefit from changing the allocation *and* the price vector.
Let $v(P, Q)$ be the maximal surplus in a market with set of buyers $P$ and set of sellers $Q$.

**Theorem**

For all $i \in P$, $\bar{u}_i = v(P, Q) - v(P \setminus \{i\}, Q)$.

This looks a lot like a mechanism you are/should be familiar with...
Fix a buyer $i \in P$ and let $j$ be such that $x_{ij} = 1$ for some optimal assignment $x$ (might have $j = O$).

Rewriting buyer-optimal stable payoff:

$$\bar{u}_i = \alpha_{ij} + [v(P, Q) - \alpha_{ij} - v(P \setminus \{i\}, Q)].$$

This is precisely buyer $i$’s payoff in the VCG mechanism. Hence, we obtain the following:

**Corollary**

*The multiobject auction (as a direct mechanism) is strategyproof for buyers.*
Summary

- In short: Everything is just as in the marriage model...
- A bit more elaborate:
  - Stable outcomes always exist in the assignment game.
  - Stability is “equivalent” to the concept of competitive equilibrium.
  - Generalization of second-price auction finds minimum competitive price/buyer-optimal stable outcome.
  - Buyer-optimal stable mechanism is strategyproof for buyers.